Mathematical

Reviews

Edited by

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R. P. Boas, Jr., *Executive Editor*

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Mathematical Reviews

FOUNDATIONS

Sesmat, Augustin, et Lalan, Victor. Élimination dans le corps de Boole et syllogisme. C. R. Acad. Sci. Paris **224**, 1411–1413 (1947).

Given two Boolean relations between p, q and q, r which are expressed by equations or inequations, the authors find (by the elimination of g) a relation between p and r necessary and sufficient for the existence of a nonconstant q consistent with the premises. They exhibit in a table the results for the 64 possible cases, 32 of which are inconclusive. The other 32 include the traditional modes of the syllogism. *O. Frink* (State College, Pa.).

Fitch, Frederic B. An extension of basic logic. J. Symbolic Logic **13**, 95–106 (1948).

The author has considered previously [same J. **7**, 105–114 (1942); **9**, 57–62, 89–94 (1944); these Rev. **4**, 125; **6**, 197; **7**, 45] a logical calculus K, within which is definable every constructively definable system of logic. It did not contain negation, universal quantification, nor the dual to ancestrality. The present paper concerns a calculus K', which beside the expressions of K admits these three basic terms. Twenty-five rules (not entirely mutually independent) are given which serve as postulates for K'. Consistency of K' is proved in that it is shown that there is in K' no U-expression "a," such that "$\sim a$" also is in K'. The notion of general recursive relation is generalized to "recursively definite relation," and it is shown for example that every recursively enumerable class of natural numbers is recursively definite. The author outlines a theory of real numbers. This theory admits of addition, multiplication, exponentiation, upper bounds and lower bounds. However, all classes considered remain enumerable. Successive extensions proposed retain this property. *A. A. Bennett* (Providence, R. I.).

Carruccio, Ettore. Alcune conseguenze di un risultato del Gödel e la razionalità del reale. Atti Soc. Nat. Mat. Modena **78**, 88–90 (1947).

Using Gödel's incompleteness theorem the author defines, corresponding to a formal logical system of the type considered by Gödel, a number r which is rational but cannot be proved to be rational by means which can be formalized within the system if the latter is consistent. Successive digits of the decimal expansion of a certain real number α are required to coincide with those of the expansion of $\sqrt{2}$ as long as no contradiction appears in an enumeration of the theorems of the formal system; otherwise the expansion terminates. Since the system is consistent, but by Gödel's theorem cannot be proved so within the system, it follows that $\alpha^2 = \alpha\sqrt{2}$ has the properties required of the rational number r. This provides a simple example of an undecidable proposition. *O. Frink* (State College, Pa.).

Carruccio, Ettore. Sull'impossibilità di esprimere integralmente in simboli un sistema ipotetico-deduttivo. Atti Soc. Nat. Mat. Modena **78**, 91–92 (1947).

The author argues that it is impossible to represent a deductive logical system having both axioms and rules of procedure entirely by means of a formalism based on a finite number of symbols, since either a vicious circle or an infinite regression necessarily results from an attempt to express the rules of procedure entirely in terms of the symbols of the formalism. This observation that rules of procedure are indispensable in a deductive system and cannot be replaced by axioms was first made by Lewis Carroll [Mind (N.S.) **4**, 278–280 (1895)]. *O. Frink* (State College, Pa.).

Gödel' [Gödel], K. The consistency of the axiom of choice and of the generalized continuum hypothesis with the axioms of set theory. Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 96–149 (1948). (Russian)

A translation by A. A. Markov of the author's book [Annals of Mathematics Studies, no. 3, Princeton University Press, 1940; these Rev. **2**, 66].

Court, N. A. Is mathematics an exact science? Scientific Monthly **67**, 119–123 (1948).

Schaaf, William L., editor. Mathematics: Our Great Heritage. Essays on the Nature and Cultural Significance of Mathematics. Harper & Brothers, New York, 1948. xi+291 pp. $3.50.

The essays, reprinted from various sources, are by J. W. N. Sullivan, G. H. Hardy, J. B. Shaw, E. T. Bell, G. Sarton, D. J. Struik, C. V. Newsom, C. G. Hempel, T. Dantzig, T. Fort, J. W. Lasley, Jr., R. B. Lindsay, T. C. Fry, A. Henderson, A. Dresden, R. D. Carmichael, and a report of the Progressive Education Association.

ALGEBRA

Ghurye, S. G. A characteristic of species of 7×7 Latin squares. Ann. Eugenics **14**, 133 (1948).

The author derives from every Latin square a scheme termed the intercalate analysis. Squares belonging to the same species must have the same intercalate analysis. The author observes that among those 7×7 squares enumerated by Norton [same Ann. **9**, 269–307 (1939); these Rev. **1**, 199] which have intercalates any two different species have different intercalate analyses. From this he conjectures that with a suitable generalization of the concept of intercalates it will be possible to characterize every species by its intercalate analysis. Since, however, Norton's enumeration is not with certainty known to be complete, the author's conjecture cannot, even for the 7×7 squares, be regarded as verified.

H. B. Mann (Columbus, Ohio).

Staver, Tor B. On summation of powers of binomial coefficients. Norsk Mat. Tidsskr. **29**, 97–103 (1947). (Norwegian)

An elementary treatment of sums of the type

$$\sum_{\nu=0}^{n}(-1)^{\nu}\nu^{p}\binom{n}{\nu}^{q} \quad \text{and} \quad \sum_{\nu=0}^{n}\nu^{p}\binom{n}{\nu}^{q}$$

(p and q integers). The author derives many recurrence formulae for fixed values of q, especially for $q=3, 2, 4, -1, -2$, e.g., $n^{2}S_{n}=(7n^{2}-7n+2)S_{n-1}+8(n-1)^{2}S_{n-2}$. In this way he obtains among others several well-known results, e.g.,

$$\sum_{\nu=0}^{2n}(-1)^{\nu}\binom{2n}{\nu}^{3}=(-1)^{n}(3n)!/(n!)^{3}$$

[A. C. Dixon, Messenger of Math. **20**, 79–80 (1890)];

$$\sum_{\nu=0}^{n}\binom{n}{\nu}^{2}=\binom{2n}{n}; \quad \sum_{\nu=0}^{2n}(-1)^{\nu}\binom{2n}{\nu}^{2}=(-1)^{n}\binom{2n}{n}.$$

S. C. van Veen (Delft).

Varoli, Giuseppe. Sulla determinazione delle radici di una equazione algebrica a coefficienti razionali. Matematiche, Catania **1**, 147–149 (1946).

Lee, H. C. Isotropic pseudo-orthogonal transformations. Quart. J. Math., Oxford Ser. **19**, 81–89 (1948).

An isotropic pseudo-orthogonal matrix A of order n is characterized by the conditions: (1) $A'(I_{p}\dotplus(-I_{q}))A=I_{p}\dotplus(-I_{q})$, $p+q=n$; (2) $Ax-x$ has the length zero for every vector x. The main result of this paper is the theorem: in the real field A can be written

$$A=I_{n}+(U\dotplus V)\bar{A}(U'\dotplus V'),$$

$$\bar{A}=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C & D & 0 \\ 0 & D & -C & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where C is the antisymmetric matrix

$$C=b_{1}\begin{pmatrix} 0 & I_{\rho_{1}} \\ -I_{\rho_{1}} & 0 \end{pmatrix}\dotplus\cdots\dotplus b_{s}\begin{pmatrix} 0 & I_{\rho_{s}} \\ -I_{\rho_{s}} & 0 \end{pmatrix}$$

of order $r=2\rho_{1}+\cdots+2\rho_{s}$, the b being real and positive numbers such that $b_{1}<b_{2}<\cdots<b_{s}$; D is the diagonal matrix $D=b_{1}I_{2\nu_{1}}\dotplus\cdots\dotplus b_{s}I_{2\nu_{s}}$, of order r; U and V are real orthogonal matrices of orders p and q respectively; and the partition in $\bar{A}$ is such that its four diagonal blocks 0, C, $-C$, 0 have orders $p-r$, r, r, $q-r$, respectively. From this theorem some consequences are drawn. A corresponding theorem for isotropic pseudo-unitary matrices is also stated.

S. Chern (Nanking).

Lee, H. C. On the factorisation of orthogonal transformations into symmetries. Bull. Amer. Math. Soc. **54**, 558–559 (1948).

A short proof is given of the theorem: every orthogonal transformation on n variables is decomposable into the product of a number not greater than n of symmetries. In the inductive proof it is necessary to consider separately the case in which $a_{1}^{1}=1$, where $A=(a_{i}^{j})$ is orthogonal. Here it is asserted that A can be transformed by another orthogonal matrix so as to render $a_{1}^{1}\neq 1$. This is false if and only if $A=I+S$, where S is skew-symmetric and $S^{2}=0$. Since such an $S\neq 0$ cannot exist in the real field, the proof is easily completed in this case. Over the complex field, the assertion is proved false by writing down such a matrix S,

which can be done if $n\geq 4$. That the theorem is actually true over the complex field was proved by P. F. Smith [Trans. Amer. Math. Soc. **6**, 1–16 (1905)]. Its validity for all fields appears to remain in doubt, although the weaker theorem that $2n$ symmetries will suffice was proved for all fields not of characteristic two by the reviewer [same Bull. **46**, 81–85 (1940); these Rev. **1**, 195]. *W. Givens.*

Urabe, Minoru. Spin transformations. I. J. Sci. Hirosima Univ. Ser. A. **11**, 171–199 (1942).

Given 4×4 matrices γ_{i}, $i=1, 2, 3, 4$, which satisfy $\gamma_{(i}\gamma_{j)}=\delta_{ij}I$, and a real orthogonal matrix $\|a_{j}^{i}\|$, the author computes the 4×4 matrix S which satisfies $\gamma_{i}=S^{-1}a_{i}^{j}\gamma_{j}S$. This is a special case of a result of A. S. Eddington [J. London Math. Soc. **7**, 58–68 (1932)]. The author then compares his work with that of R. Brauer and H. Weyl [Amer. J. Math. **57**, 425–449 (1935)] where the matrices S were constructed by an infinitesimal method, extends his result to 8×8 matrices, and shows that the same procedure can be extended to $2^{n}\times 2^{n}$ matrices. There are many typographical errors. *A. Schwarz* (New York, N. Y.).

Abstract Algebra

Plechl, O., und Duschek, A. Grundzüge einer Algebra der elektrischen Schaltungen. Österreich. Ing.-Arch. **1**, 203–230 (1946).

A correspondence is set up between systems of electrical switches and the expressions of a certain algebra, in such a way that algebraically equivalent expressions correspond to systems of switches which are electrically equivalent, though they may be physically different. The algebra is shown to be identical with the ordinary algebra of logic (Boolean algebra). It may be used to simplify a given switching system, or to synthesize one having preassigned properties. This is illustrated by examples. The authors define standard form, normal form, and reduced form of a Boolean expression. They seem to be unacquainted with the earlier results of C. E. Shannon along the same lines [Trans. Amer. Inst. Elec. Engrs. **57**, 713–723 (1938)]. *O. Frink.*

Monteiro, Antonio. Réticulés distributifs de dimension linéaire N. C. R. Acad. Sci. Paris **226**, 1658–1660 (1948).

The arithmetic of the prime dual ideals ["filtres premiers"; cf. the author's paper, same C. R. **225**, 846–848 (1947); these Rev. **9**, 265] of a lattice L is said to be of order n if n is the smallest number such that each dual ideal of L is the intersection of not more than n minimal prime dual ideals. The author states the following results in case L is distributive: (1) this order n equals the linear dimension of L (least number of chains whose product contains L); (2) this order n equals the smallest n for which $n+1$ elements of L are always dependent. *P. M. Whitman.*

Châtelet, François. Formes quadratiques dans un corps arbitraire. C. R. Acad. Sci. Paris **226**, 1233–1235 (1948).

Let Q be a quaternary quadratic form with coefficients in a field K. If it is indecomposable it represents a quadric surface. A homography exists relating the rulings of this quadric to the points of two conics γ_{1}, γ_{2} which are conjugate in the field $K(\sqrt{d})$, where d is the discriminant of Q. The problem of the classification of quaternary quadratic forms in a field K and of discriminant d is thus reduced to the

[handwritten note at bottom of page:] In noting an error in the paper the reviewer stated that the validity of the author's theorem for an arbitrary field "appears to remain in doubt." This is not now true, a proof having been published by J. Dieudonné [p. 20 of the book reviewed in these Rev. 9, 494–495]

Errata p. 255, v.10, these Rev.

classification of ternary forms in $K(\sqrt{d})$. Moreover the determination of the zeros of Q in K is reduced to the determination of the zeros of γ_1 and γ_2 in $K(\sqrt{d})$. This throws new light on the criterion of Witt [J. Reine Angew. Math. 176, 31–44 (1936)].   *C. C. MacDuffee* (Madison, Wis.).

Durfee, William H. Quadratic forms over fields with a valuation. Bull. Amer. Math. Soc. **54**, 338–351 (1948).

L'auteur généralise les résultats de Hasse sur l'équivalence des formes quadratiques à coefficients p-adiques au cas des formes à coefficients dans certains corps valués complets; une partie de ces résultats subsistent dans tous les corps valués complets dont le corps de restes n'est pas de caractéristique 2. Soient K un corps valué complet, T son domaine d'intégrité, U son groupe multiplicatif des unités, $K^{(2)}$ le groupe multiplicatif des carrés des éléments non-nuls de K, R (supposé de caractéristique non 2) son corps de restes, $\mathfrak{M}$ son module de valuation. Le reste d'un $a \varepsilon T$ suivant l'idéal premier de K sera noté $\bar{a}$. Des éléments a, $b \varepsilon K$ seront dits congrus (notation: $a \equiv b$) s'ils le sont suivant le groupe multiplicatif $UK^{(2)}$. Soit $f = \sum a_i x_i^2$ une forme quadratique à coefficients a_i dans K, mise sous une forme diagonale (ce qui est un problème banal). Donc $\bar{f} = \sum \bar{a}_i x_i^2$ ($a_i \varepsilon T$) est dite le reste de f, et f est dite unitaire si tous les a_i sont des unités de K. Elle est dite une forme standartisée, si $f = \sum b_i f_i$, où les f_i sont des formes unitaires et où les b_i sont incongrus deux à deux. Chaque forme de K est équivalente à une forme standartisée. Si f représente 0 dans K ou dans T, f elle est dite nullique (zero form) dans K ou dans T; f est dite totalement nullique et est notée H_q si

$$f = (x_1^2 - x_2^2) + (x_3^2 - x_4^2) + \cdots + (x_{q-1}^2 - x_q^2).$$

L'auteur montre que si $f = \sum b_i f_i$ est standartisée, elle est nullique dans K si, et seulement si, une des f_i l'est, et que, si $g = \sum c_i g_i$ est aussi standartisée, la question d'équivalence dans K des f, g se réduit à celle de certaines de leurs composantes f_i, g_j ou de leurs sommes avec des formes totalement nulliques convenables, donc à l'équivalence des formes unitaires. Ensuite, l'auteur prouve qu'une forme unitaire est nullique dans K ou deux formes unitaires f, g y sont équivalentes si, et seulement si cela a lieu pour ces formes dans T, et que cela a lieu dans T si, et seulement si cela a lieu dans R pour les restes de ces formes.

Soient $|f|$ le déterminant de f, $(a, b)[a, b \varepsilon K]$ le symbole égal à $+1$ ou -1 suivant que la forme $ax^2 + by^2$ représente ou non 1 dans K, $c(f) = \prod_{i=1}^{n}(-d_{i-1}, d_i)$, où $d_0 = 1$, et où $d_i = a_1 a_2 \cdots a_i$. Sous les hypothèses simultanées suivantes: (1) K est discrètement valué [mais il me semble qu'il suffit de supposer $(\mathfrak{M} : 2\mathfrak{M}) = 2$]; (2) $(R^* : R^{(2)}) = 2$ (où R^* est le groupe multiplicatif des éléments non nuls de R); (3) pour tous les α, $\beta \varepsilon R^*$, $(\alpha, \beta) = +1$, l'auteur démontre les résultats suivants, dus à Hasse dans le cas p-adique: f est nullique si, et seulement si $n = 2$ et $-|f| \varepsilon K^{(2)}$, ou $n = 3$ et $c(f) = 1$, ou $n = 4$ et $c(f) = 1$ ou $-|f| \varepsilon K^{(2)}$, ou $n \geq 5$; deux formes non-singulières f, g d'un même nombre de variables sont équivalentes si, et seulement si $|f| = |g|$ $(\mathrm{mod}^\times K^{(2)})$ et $c(f) = c(g)$. En particulier, si ν est une unité de K telle que $\bar{\nu} \not\equiv 1 \pmod{^\times R^{(2)}}$, et si t est un élément de K dont l'ordre n'est pas dans $2\mathfrak{M}$, f équivaut toujours à une forme canonique $x_1^2 + x_2^2 + \cdots + x_{s-1}^2 + ax_s^2 + t(x_{s+1}^2 + \cdots + x_{n-1}^2 + bx_n^2)$, où $s \leq n$ et où a, b peuvent prendre indépendamment les valeurs 1, ν.   *M. Krasner* (Paris).

Carruth, Philip W. Generalized power series fields. Trans. Amer. Math. Soc. **63**, 548–559 (1948).

Let the field K be maximal in the valuation V, with value group Γ and residue class field $\mathfrak{K}$. In case K and $\mathfrak{K}$ have the same characteristic, Kaplansky has shown that K is analytically isomorphic to a power series field if either: (1) Γ and $\mathfrak{K}$ satisfy a certain hypothesis A [Duke Math. J. 9, 303–321 (1942); these Rev. 3, 264]; or (2) Γ is discrete [Duke Math. J. 12, 243–248 (1945); these Rev. 7, 3]. In the present paper the structure of K is studied when either of these conditions is satisfied, but when the characteristics of K and $\mathfrak{K}$ are not necessarily the same. In particular it is shown that, when the characteristic p of $\mathfrak{K}$ is different from that of K, K is analytically isomorphic to a generalized power series field if either: (1) Γ and $\mathfrak{K}$ satisfy hypothesis A, and Γ is the isolated subgroup of itself generated by $V(p)$; or (2) Γ is Archimedean and discrete. A further result is that the cardinal number of any maximal field K is equal to that of Hahn's power series field $K(x^\Gamma)$ [Akad. Wiss. Wien, S.-B. IIa. 116, 601–655 (1907)].   *B. N. Moyls.*

Kolchin, E. R. Algebraic matric groups and the Picard-Vessiot theory of homogeneous linear ordinary differential equations. Ann. of Math. (2) **49**, 1–42 (1948).

The Picard-Vessiot theory plays with respect to linear homogeneous differential equations the same role as Galois theory with respect to algebraic equations. Let F be a differentiable field of characteristic 0 whose field of constants C is algebraically closed. By a Picard-Vessiot extension G of F is meant a differentiable field of the form $G = F\langle y_1, \cdots, y_n \rangle$, where $(y_1, \cdots, y_n)$ is a fundamental system of solutions of a linear differential equation $L(y) = 0$ with coefficients in F, and where it is assumed that G has the same field of constants C as F. The Picard-Vessiot theory establishes a relationship between the structure of the group H of automorphisms of G over F and the nature of the analytic operations which are necessary to obtain $y_1, \cdots, y_n$. Roughly speaking, the main result of the theory is that the equation $L(y) = 0$ can be solved by quadratures if and only if the group H is solvable. The author observes that the words "solvable by quadratures," as used by Picard and Vessiot, do not have a precisely defined meaning: it is not said whether they include the taking of the exponential of an integral, or the adjunction of algebraic irrationalities. In fact, it turns out that the proof of the necessary and sufficient character of the criterion of solvability is vitiated by the fact that one meaning of the words "solvable by quadratures" is used in the necessity part, and another in the "sufficiency" part. By allowing various combinations of permissible operations (adjoining integrals, exponentials of integrals, algebraic irrationalities, radicals), the author distinguishes ten different cases of solvability by quadratures and he characterizes each of these ten cases by a suitable property of the group H.

The basic fact is that the group H may be represented as a group of matrices of degree n with coefficients in C, and that this group is an algebraic group; i.e., the property that a regular matrix belongs to the group may be expressed by a system of algebraic equations which the coefficients must satisfy. In the classical theory this fact was made use of through its immediate consequence to the effect that H is a Lie group (provided C is the field of complex numbers), and the machinery of Lie theory was used extensively. How-

ever, this procedure was unsatisfactory for two reasons: first methodologically, because it is unfortunate to bring into this algebraic question the transcendental methods of Lie theory; second, because the local character of the classical Lie theory excludes the consideration of the global properties of the group H (for instance, its connectedness) which play an essential role in the Kolchin classification. For these reasons, the author first develops an autonomous theory of algebraic groups of matrices, which is mainly oriented towards the study of solvable groups. In particular, the theorem of Lie on the reducibility of solvable groups of matrices is proved (for algebraic groups) without any assumption on the characteristic of the basic field. On the other hand, the author introduces the following new properties of algebraic groups of matrices H: H is called anticompact if it contains no other element than the identity whose order is not divisible by the characteristic of the basic field, and quasicompact if no algebraic subgroup of order greater than 1 is anticompact. A necessary and sufficient condition for H to be anticompact (quasicompact) is for every matrix in H to have its characteristic roots all equal to 1 (to be reducible to diagonal form). The characters of anticompacity and quasicompacity play an important role in the classification of various cases of solvability by quadratures in the theory.   *C. Chevalley* (Princeton, N. J.).

Mori, Shinziro. Über Ringe, die den Durchschnittssatz gestatten. J. Sci. Hirosima Univ. Ser. A. **11**, 129–136 (1942).

Let $\Re$ be a commutative ring in which: (1) every ascending chain of prime ideals is finite; (2) every chain of ideals of the form $\mathfrak{a} \subset \mathfrak{a}:\mathfrak{b}_1 \subset \mathfrak{a}:\mathfrak{b}_1\mathfrak{b}_2 \subset \cdots$ is finite. It is proved that under these conditions every ideal is an intersection of finitely many strongly primary ideals. The author states erroneously that no necessary and sufficient condition is known for this decomposition to hold in a ring. Such a condition has been given by Krull [S.-B. Heidelberger Akad. Wiss. **1929**, no. 2, 11–16 (1929); Idealtheorie, Springer, Berlin, 1935, p. 17], who also considers decomposition into weakly primary ideals. The reviewer notes that a slight modification of Krull's proof yields a simpler proof of the present result than that given by the author.
                    I. S. Cohen (Cambridge, Mass.).

Mori, Shinziro. Representation of ideals as intersections of weak primary ideals. J. Sci. Hirosima Univ. Ser. A. **12**, 1–10 (1942). (Japanese)

The following summary is taken from the author's abstract in Jap. J. Math. **18**, abstracts, p. 11 (1943). Cf. also the preceding review. Es sei $\Re$ ein kommutativer Ring, in dem jede Kette von Primidealen $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3 \subset \cdots$ nach endlich vielen Gliedern abbricht. Die notwendige und hinreichende Bedingung dafür, dass in $\Re$ jedes Ideal sich als Durchschnitt von endlich vielen schwachen Primäridealen darstellen lässt, besteht darin, dass in $\Re$ die folgenden Bedingungen erfüllt sind. (1) Ist ein Element r nicht nilpotent in bezug auf ein Ideal $\mathfrak{a}$, so müssen von einem gewissen n ab alle Glieder der Kette $\mathfrak{a} \subset \mathfrak{a}:(r) \subset \mathfrak{a}:(r^2) \subset \cdots$ einem Ideal τ gleich sein. (2) Für ein Ideal $\mathfrak{a}$ ist die Anzahl der obig gewonnenen Ideale τ endlich.

Kaplansky, Irving. Locally compact rings. Amer. J. Math. **70**, 447–459 (1948).

The results and methods (in particular the concepts of boundedness, quasi-inverse, Q-rings, etc.) of a recent paper [same J. **69**, 153–183 (1947); these Rev. **8**, 434] by the author on topological rings are here applied to locally compact rings. Locally compact rings have been treated by Jacobson and Taussky [Proc. Nat. Acad. Sci. U. S. A. **21**, 106–108 (1935)] and others subsequently. Some of the earlier results (in the connected case) are obtained here as special cases of a single theorem. In the general case, if no (algebraically) nilpotent ideals exist the ring is the direct sum of a finite hyper-complex system (with real coefficients) and a totally disconnected ring. For bounded semi-simple locally compact rings a decomposition into a compact and a discrete ring is exhibited. The structure of both types, if they also satisfy the descending chain condition, is given in a unified treatment. The existence and continuity of the inverse is then studied. For totally disconnected locally compact rings a system of neighborhoods of 0 is shown which are compact open subrings and the structure theory for compact rings is then applied. A connected locally compact ring is shown to be a Q-ring with continuous quasi-inverse. The question raised in the earlier paper as to whether any Q_r-ring is a Q-ring is answered affirmatively in the locally compact case. By the same method several theorems of that paper are now proved without countability assumptions and Otobe's result [Jap. J. Math. **19**, 189–202 (1945); these Rev. **7**, 237] that the inverse is continuous in locally compact division rings is generalised in several ways. However, an example of a locally compact ring without continuity of the inverse is given. From the existence of compact open subrings several results concerning maximal ideals are obtained, particularly closed maximal ideals. The latter provide a partial reduction of locally compact semi-simple rings to primitive ones. Several new results concerning primitive rings are also given.
                    O. Todd-Taussky (London).

Uzkov, A. I. An algebraic lemma and the normalization theorem of E. Noether. Mat. Sbornik N.S. **22(64)**, 349–350 (1948). (Russian)

The normalization theorem of E. Noether states the following: if $R = K[u_1, \cdots, u_n]$ is a finite integral domain over a field K and if m is the transcendence degree of R over K, then there exist m elements $v_1, \cdots, v_m$ in R such that R is integrally dependent on the ring $K[v_1, \cdots, v_m]$. This theorem, proved by E. Noether for infinite fields K, has been extended to arbitrary fields by the reviewer [Trans. Amer. Math. Soc. **53**, 490–542 (1943); these Rev. **5**, 11].

The author's more direct and elementary proof is based on the lemma: if $f(X_1, \cdots, X_n)$ is a nonzero polynomial with coefficients in K, then there exist integers $h_1, \cdots, h_{n-1}$ such that the substitution $X_n = X_n'$, $X_i = X_i' + X_n'^{h_i}$, $i = 1, \cdots, n-1$, transforms $f(X)$ into a polynomial $f'(X')$ of the form $a_0 X_n'^s + A_1(X_1', \cdots, X_{n-1}') X_n'^{s-1} + \cdots + A_s(X_1', \cdots, X_{n-1}')$, where $a_0 \in K$, $a_0 \neq 0$ and the A_j are polynomials with coefficients in K. This lemma is proved by induction with respect to n. From the lemma it follows that if the transcendence degree m of R/K is less than n (in the case $m = n$ there is nothing to prove), then there exists another set of ring generators $u_1', \cdots, u_n'$ of R over K (here $u_n' = u_n$) such that u_n' is integrally dependent on $K[u_1', \cdots, u_{n-1}']$, and this gives the normalization theorem at once.
                    O. Zariski (Cambridge, Mass.).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: *f*-extensions; conducteur. C. R. Acad. Sci. Paris **226**, 1231–1233 (1948).

Krasner, Marc. Théorie non abélienne des corps de classes pour les extensions galoisiennes des corps de nombres algébriques: forme définitive de la loi d'existence. C. R. Acad. Sci. Paris **226**, 1656–1658 (1948).

If k is a field and $f(x)$ an irreducible polynomial of the ring $K[x]$ of polynomials over k, the f-extension $K^{(f)}$ of k is simply the residue class field of $k[x]$ modulo the prime ideal generated by $f(x)$. It is of course isomorphic to the algebraic extension of k by a root of $f(x)=0$, and $K^{(f)}$ and $K^{(g)}$ are isomorphic if $f(x)$ and $g(x)$ define the same algebraic field over k.

Now if k is an algebraic number field and $\bar{k}$ its $\mathfrak{p}$-adic completion with respect to a prime ideal $\mathfrak{p}$ it is well known that the prime ideal divisors of $\mathfrak{p}$ in the extended field $K^{(f)}$ correspond to the irreducible factors of $f(x)$ in $\bar{k}$. Let $\bar{f}(x)$ be any prime factor of $f(x)$ in $\bar{k}$ and let $\mathfrak{P}(\bar{f})$ be the corresponding prime ideal divisor of $\mathfrak{p}$ in $K^{(f)}$, $|\cdots|_{\mathfrak{P}(\bar{f})}$ the corresponding valuation of $K^{(f)}$, and $\bar{K}^{(f)}$ the completion of $K^{(f)}$ with respect to this valuation. If $a(x)$, or strictly its residue class modulo $f(x)$, is any element of $K^{(f)}$, the value of $|a(x)|_{\mathfrak{P}(\bar{f})}$ can be determined by a finite number of rational operations and is in fact equal to $|R[a(x), \bar{f}(x)]|^{1/\nu}$, where ν is the degree of $\bar{f}(x)$, $R[a(x), \bar{f}(x)]$ is the resultant of $a(x)$ and $\bar{f}(x)$ and $|\cdots|$ is $\mathfrak{p}$-adic valuation in $\bar{k}$. [In this connection see also J. C. Fields, Proc. Int. Math. Congress, Toronto, 1924, v. 1, pp. 245–298, and S. Beatty and D. C. Murdoch, Fundamental Exponents in the Theory of Algebraic Numbers, University of Toronto Press, 1937.]

A further discussion leads to an expression for the conductor of the field $K^{(f)}$ as a product of powers of prime ideals. The notation used and many of the concepts involved depend on the author's previous notes [same C. R. **225**, 785–787, 973–975, 1113–1115 (1947); **226**, 535–537 (1948); these Rev. **9**, 223, 326, 408].

The second note gives the characterization of the principal ring R_δ and the definitive form of the law of existence which were promised in the previous notes cited above. The problem of determining in a finite number of rational operations which representations belong to the Takagi ring $\mathfrak{T}_{K/k}$ of an extension K/k is made to depend upon the theorem of R. Brauer that every character of a group $\mathfrak{G}$ is a linear combination with rational integral coefficients of characters induced in $\mathfrak{G}$ by characters of prime degree of certain "elementary" subgroups of $\mathfrak{G}$ [R. Brauer, Ann. of Math. (2) **48**, 502–514 (1947); these Rev. **8**, 503].

D. C. Murdoch (Vancouver, B. C.).

Nakayama, Tadasi, and Azumaya, Gorô. On irreducible rings. Ann. of Math. (2) **48**, 949–965 (1947).

A ring is called (right) irreducible when it has a faithful irreducible right module; ideal-irreducible if it possesses moreover a faithful irreducible right ideal. An irreducible ring is called (right) closed if, in the endomorphism ring of some faithful right module, it is the commutator of its commutator. A (right) closed irreducible ring $\mathfrak{R}$ is right and left ideal-irreducible and is isomorphic to the ring $\mathfrak{K}_M$ of all row-finite M-dimensional matrices, where $\mathfrak{K}$ is a quasi-field and M a (possibly infinite) cardinal number. Only for finite M is it also left-closed. All minimal right-ideals of $\mathfrak{R}$ are mutually isomorphic, and their sum is the unique smallest two-sided ideal $\mathfrak{z}$ in $\mathfrak{R}$. In $\mathfrak{K}_M$, the ideal $\mathfrak{z}$ consists of the set of all matrices with only a finite number of nonzero columns. The ring $\mathfrak{R}$ is the endomorphism ring of $\mathfrak{z}$, considered as $\mathfrak{R}$-left module. [Reviewer's comment. It is well known that to generalize simple rings by dropping chain condition, but retaining existence of minimal one-sided ideals, one must give up either existence of unit element or nonexistence of proper two-sided ideals. So the above defined closed irreducible ring is one of the natural generalizations of simple ring although $\mathfrak{z}\neq(0)$ and $\mathfrak{z}\neq\mathfrak{R}$ whenever M is infinite.]

The authors study representation theory, Kronecker products and commutators of subrings; their results on the latter subject contain a complete generalization of the known theorems on simple subalgebras (over the center) of a simple ring to simple subalgebras of a closed irreducible ring. Finally, they study Galois theory of a closed irreducible $\mathfrak{R}$ with respect to a finite group G of outer automorphisms. If $a_{S,T}$ is a factor set of elements of $\mathfrak{R}$ then it defines a crossed product of $\mathfrak{R}$ and G which is again closed irreducible. By use of this crossed product for unit factor set, they prove very quickly not only the expected theorems on one-to-one correspondence between subgroups of G and closed irreducible subrings between $\mathfrak{R}$ and its subring $\mathfrak{S}$ (itself closed irreducible) left element-wise fixed by G, but also the fact that $\mathfrak{R}$ can be expressed as a full matrix ring $(\mathfrak{R}_0)_N$ of row-finite matrices with G-invariant matrix units over a simple ring with chain condition $\mathfrak{R}_0$; and that $\mathfrak{S}=(\mathfrak{S}_0)_N$ where $\mathfrak{S}_0$ is a quasi-field. Thus Galois theory for irreducible rings is in this case reduced to Galois theory for simple rings.

As far as the reviewer knows, this [together with the paper reviewed below] is the first extension of Galois theory beyond quasi-fields. Galois theory of inner automorphisms is not discussed. [Though most of these results are contained nowhere else, they have some intersection with recent papers which were evidently inaccessible to the authors, e.g., simple rings: Artin and Whaples, Amer. J. Math. **65**, 87–107 (1943); these Rev. **4**, 129; generalized simple rings: N. Jacobson, Trans. Amer. Math. Soc. **57**, 228–245 (1945); same Ann. (2) **48**, 8–21 (1947); J. Dieudonné, Bull. Soc. Math. France **70**, 46–75 (1942); these Rev. **6**, 200; **8**, 433; **6**, 144; Galois theory: N. Jacobson, Amer. J. Math. **69**, 27–36 (1947); H. Cartan, Ann. Sci. École Norm. Sup. (3) **64**, 59–77 (1947); these Rev. **9**, 4, 325.] *G. Whaples.*

Dieudonné, Jean. La théorie de Galois des anneaux simples et semi-simples. Comment. Math. Helv. **21**, 154–184 (1948).

The author extends the Galois theory of Cartan and Jacobson [cited in the preceding review] to rings (called primitive complete) which are isomorphic to the ring of all K-endomorphisms of a vector space (without restriction as to finite basis) over a quasi-field K. Since he writes endomorphisms as left operators, these primitive complete rings would be described in the language of the preceding review as the ring of all column-finite matrices over K. Direct sums of primitive complete rings (called semi-simple) are also studied. Rings are considered as subrings of the endomorphism ring $\mathfrak{E}$ of a module (without operators) E; such a ring is called admissible [distingué] if it is semi-simple, contains the unit element of $\mathfrak{E}$, and is identical with the closure of its socle [the socle coincides with the ideal $\mathfrak{z}$ of the preceding review; see J. Dieudonné, reference in the preceding review]. The commutator in $\mathfrak{E}$ of an admissible subring is also admissible.

After studying the structure of these rings and proving a generalization of known theorems on commutators in simple rings, the author sets up the following generalized Galois theory. Let A be a primitive complete ring, and E a vector space over the quasi-field K. An admissible $B \subset A$ is called Galoisian if its commutator C (in $\mathfrak{C}$) consists of linear combinations, over K, of semilinear transformations with respect to K; B is called interior-Galoisian if these transformations are linear over K. In both cases, B is strongly Galoisian if every element of C is a linear combination of automorphisms (not merely endomorphisms) of E. If E has finite basis over K, "Galoisian" implies "strongly Galoisian"; whether this is so in general is unknown.

If B is Galoisian in A, so is every admissible B' with $B \subset B' \subset A$; B is interior-Galoisian if and only if it is the commutator in A of a subring $D \subset A$ for which $D \otimes K$ (over the center of K) is admissible. If K is of finite degree over its center Z, then the commutator of every admissible $D \supset Z$ is Galoisian.

Many further results are obtained involving conditions that a subring be strongly Galoisian, that C be generated by a group of outer automorphisms, and that an isomorphism of two subrings can be extended to an automorphism. Application of this theory to give new derivations of known results on crossed products over commutative fields is discussed.   *G. Whaples* (Bloomington, Ind.).

Jacobson, N. Isomorphisms of Jordan rings. Amer. J. Math. **70**, 317–326 (1948).

In this paper, the author considers 4 classes of Jordan rings constructed from simple associative rings. These classes are as follows. (A_I) $\mathfrak{A}$ is a simple associative ring of finite dimensionality over its centre Φ. The Jordan ring $\mathfrak{A}_J$, obtained from $\mathfrak{A}$ by replacing the ordinary multiplication by the Jordan multiplication $a \cdot b = (ab + ba)$, is said to be of type A_I. (A_{II}) $\mathfrak{A}$ is as above, but the centre is now denoted by P; J is an involution of $\mathfrak{A}$ inducing a nontrivial automorphism in P; $\mathfrak{H}(\mathfrak{A}, J)$ is the subring of $\mathfrak{A}_J$ consisting of the J-symmetric elements of $\mathfrak{A}$; Φ denotes $\mathfrak{H} \cap P$. Then P is a quadratic extension of Φ and $\mathfrak{H}(\mathfrak{A}, J)$ is termed a Jordan ring of type A_{II}. (B, C) $\mathfrak{A}$ is as above, with centre Φ; J is an involution of $\mathfrak{A}$ leaving the elements of Φ invariant. In each case the Jordan ring considered is the subring $\mathfrak{H}(\mathfrak{A}, J)$ of $\mathfrak{A}_J$ consisting of the J-symmetric elements of $\mathfrak{A}$. The distinction between types B and C depends on the behaviour of J when Φ is extended to an algebraically closed field Ω. This extends $\mathfrak{A}$ to a full matrix algebra Ω_n. Then J can either extend to the mapping $a \to a'$, a' being the transpose of a, when both J and $\mathfrak{H}(\mathfrak{A}, J)$ are said to be of type B, or to the mapping $a \to q^{-1}a'q$, where

$$q = \begin{pmatrix} 0, & 1, \\ -1, & 0, \end{pmatrix}$$

(whence $n = 2\nu$). In the second case, J and $\mathfrak{H}(\mathfrak{A}, J)$ are said to be of type C.

The author proves that Φ is the centre of each of the above Jordan rings and that $\mathfrak{A}$ is the enveloping ring in each case. The dimensionalities of the various Jordan rings considered over their centres are given by:

A_I: $(\mathfrak{U}:\Phi) = n^2$, $\mathfrak{U} = \mathfrak{A}_J$, $(\mathfrak{A}:\Phi) = n^2$;

A_{II}: $(\mathfrak{U}:\Phi) = n^2$, $\mathfrak{U} = \mathfrak{H}(\mathfrak{A}, J)$, $(\mathfrak{A}:\Phi) = 2n^2$;

B: $(\mathfrak{U}:\Phi) = n(n+1)/2$, $\mathfrak{U} = \mathfrak{H}(\mathfrak{A}, J)$, $(\mathfrak{A}:\Phi) = n^2$;

C: $(\mathfrak{U}:\Phi) = n(n-1)/2$, $\mathfrak{U} = \mathfrak{H}(\mathfrak{A}, J)$, $(\mathfrak{A}:\Phi) = n^2$.

By consideration of dimensionality and the degree (the maximum dimension of subalgebras over Φ generated by

single elements) of the various Jordan rings, it is shown that rings of types A, B, and C cannot be isomorphic if they are of different types. Finally, using results of Ancochea [Ann. of Math. (2) **48**, 147–153 (1947); these Rev. **8**, 310], and Kalisch [Trans. Amer. Math. Soc. **61**, 482–494 (1947); these Rev. **8**, 561], it is shown that rings of type A_{II} cannot be isomorphic to rings of type A_I and the following results are established. (A_I) $\mathfrak{A}_J$ and $\mathfrak{B}_J$ are isomorphic if and only if $\mathfrak{A}$ and $\mathfrak{B}$ are either isomorphic or anti-isomorphic. The automorphisms of $\mathfrak{A}_J$ are those induced by the automorphisms and anti-automorphisms of $\mathfrak{A}$. (A_{II}, B, C) If $\mathfrak{H}(\mathfrak{A}, J), \mathfrak{H}(\mathfrak{B}, K)$ are isomorphic, so are $\mathfrak{A}$ and $\mathfrak{B}$. A necessary and sufficient condition for $\mathfrak{H}(\mathfrak{A}, J)$ and $\mathfrak{H}(\mathfrak{A}, K)$ to be isomorphic is that J and K should be cogredient, i.e., there exists an automorphism S of $\mathfrak{A}$ such that $K = S^{-1}JS$. Further, the automorphisms of $\mathfrak{H}(\mathfrak{A}, J)$ are those induced by automorphisms of $\mathfrak{A}$ commuting with J.   *D. Rees* (Manchester).

Andrunakievič, V. A. Semiradical rings. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 129–178 (1948). (Russian)

[Some of the results were announced in C. R. (Doklady) Acad. Sci. URSS (N.S.) **55**, 3–5 (1947); these Rev. **8**, 560.] In any ring we may introduce the operation $a \circ b = a + b - ab$ and regard it as a new multiplication. It is not quite distributive, but there is always an identity element 0; a unit element 1 plays a singular role similar to that of 0 for multiplication, but of course such an element need not be present. The analogue of a ring without divisors of 0 is a semiradical ring, in which the cancellation law holds with respect to $\circ$; the analogue of a division ring is a radical ring, in which the elements form a group under $\circ$. This parallel theory is explored by the author with complete detail. The parallelism is indeed so far-reaching that virtually all the proofs entail only minor modifications of the corresponding proofs for ordinary multiplication. Occasionally there are even simplifications, notably in the embedding problem, which is not encumbered by the necessity of excluding a singular element from the denominator.

The main theorems will now be summarized. A result due to Baer [Bull. Amer. Math. Soc. **48**, 630–638 (1942); these Rev. **4**, 70] is generalized by showing that, under suitable chain conditions, $a \circ b = 0$ and $b \circ a = 0$ are equivalent. The descending chain condition on ideals $A(1-x)$ implies that a semiradical ring A is radical, and the descending chain condition on principal left ideals makes A a nil ring. Conditions are found for A/I to be a radical ring, analogous to I's being a maximal ideal. The embedding of a semiradical ring in a radical ring is then taken up; following Ore [Ann. of Math. (2) **32**, 463–477 (1931)], it is carried out on the assumption that any two elements have a right common $\circ$-multiple. Malcev's example [Math. Ann. **113**, 686–691 (1937)] is adapted to show that the embedding is not always possible. The simplified embedding in the commutative case is presented. The author calls I a quasi-ideal [reviewer's translation] in the commutative ring A if it is closed under the operation $a - b + c$ and $a \in I$ implies $r \circ a \in I$ for all $r \in A$. (If A has a unit element 1, a quasi-ideal is obtained by adding 1 to the elements of an ideal.) Uniqueness of $\circ$-factorization is proved for semiradical rings in which every quasi-ideal is principal (an example is the ring of even integers). Finally, uniqueness of factorization of quasi-ideals as products of prime quasi-ideals is proved on the basis of the analogues of E. Noether's three axioms.

    I. Kaplansky (Princeton, N. J.).

Good, R. A. On the theory of clusters. Trans. Amer. Math. Soc. 63, 482–513 (1948).

A "cluster" is a ring with non-Abelian addition and non-associative multiplication. Numerous examples are given, all synthetic. Homomorphisms and ideals are studied. The set of all annihilators is an ideal. The "derived ring" is the set of all factorable elements. Addition among these elements is necessarily Abelian. A construction is given for extending a ring to a cluster by means of a given set Γ. Effectively Γ is a set of representatives, one from each coset of the cluster with respect to the ring (as an additive group). Fourteen conditions are given on the ring and the set Γ which are shown to be both necessary and sufficient that this construction be possible. As a corollary an answer to the following group theory problem is found. Given a group Σ and a set Γ, to extend the group Σ to a larger group Λ, so that a set of representatives of the cosets of Λ with respect to Σ is isomorphic to Γ. Note that Σ is not necessarily normal in Λ.

The paper ends with some results on finite clusters, the number of elements of which is the product of at most two primes. If the order is a prime π, the cluster is the prime finite field $GF(\pi)$, or it is trivial (all products zero). If the order is the square of a prime, the cluster is a ring. If the order is the product of distinct primes, $\pi\rho$, there are two cases. If $\rho \not\equiv 1 \pmod{\pi}$, then the cluster is a ring. If $\rho \equiv 1 \pmod{\pi}$ then there are exactly $\pi - 1$ nontrivial clusters, for each of which the derived ring is $GF(\pi)$.   *H. Campaigne.*

THEORY OF GROUPS

Fumi, F. Rappresentazione analitica dei reticoli cristallini di traslazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 370–375 (1947).

Fumi, F. Celle elementari di Bravais e traslazioni primitive di Seitz. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 376–380 (1947).

Im Anschluss an seine beiden früheren Arbeiten [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 101–109, 109–114 (1947); diese Rev. 9, 174] stellt der Verf. die zu den Bravais Gittern gehörenden Translationen auf.
    J. J. Burckhardt (Zürich).

Götlind, Erik. Some theorems on groups of order $p^r q^s$. Norsk Mat. Tidsskr. 30, 11–16 (1948). (Swedish)

Generalising a result of a previous paper [same Tidsskr. 28, 13–16 (1946); these Rev. 7, 511] the author proves that if $p^r > q^s$ and $p \not\equiv 1 \pmod{q}$, the number of non-Abelian groups of order $p^r q^s$ (p, q primes) which have a cyclic Sylow group of order p^r is equal to the number of non-Abelian groups of order q^s. He then considers groups of order $p^2 q^s$ ($p > q^s$, $p \not\equiv 1 \pmod{q}$), whose Sylow group U of order p^2, with elements A_i, is not cyclic; and shows that for any element B of a Sylow group of order q^s, either B commutes with every A_i; or else (i) B commutes with no A_i, (ii) A_i and A_j ($=BA_iB^{-1}$) generate U, (iii) $BA_jB^{-1}=A_i{}^u A_j{}^v$, where u and v satisfy certain congruences.   *D. E. Rutherford.*

Gol'fand, Yu. A. On groups all of whose subgroups are special. Doklady Akad. Nauk SSSR (N.S.) 60, 1313–1315 (1948). (Russian)

Miller and Moreno [Trans. Amer. Math. Soc. 4, 398–404 (1903)] investigated the problem of finding those finite non-Abelian groups for which every proper subgroup is Abelian. Šmidt [Rec. Math. [Mat. Sbornik] 31, 366–372 (1924)], in a more general investigation, found that all the (finite) groups of type S are of an order $p^\alpha q^\beta$, where p and q are primes. A group of type S is a nonspecial group, every proper subgroup of which is special, where a special group is one which is the direct product of cyclic groups of prime power orders. In the present paper it is found that, for fixed p, q and α, there exists essentially one group Γ_0 of type S with maximal order $p^\alpha q^{\beta_0}$, where $\beta_0 = b$ if b is odd and $\beta_0 = \frac{1}{2}b$ if b is even, where b is the least positive integer for which $q^b \equiv 1 \pmod{p}$. For fixed p, q and α, all other groups of type S may be obtained from Γ_0 by reducing it with respect to its central normal divisors. Let Σ be the subgroup of order q^β of a group Γ of type S with order $p^\alpha q^\beta$; and let Φ be the largest normal divisor of Γ included in Σ. Typical of the author's results are the following. If Σ is non-Abelian, every element of Φ has order q, the same result holding for elements of Σ if $q \not= 2$. For non-Abelian Σ, Φ is also the center and first derivative of Σ.   *F. Haimo* (St. Louis, Mo.).

Gol'fand, Yu. A. On an isomorphism between extensions of groups. Doklady Akad. Nauk SSSR (N.S.) 60, 1123–1125 (1948). (Russian)

The author points out that the construction of Schreier [Monatsh. Math. Physik 34, 165–180 (1926); Abh. Math. Sem. Univ. Hamburg 4, 321–346 (1926)] for all the extensions of a group A by a group B may yield extensions which are seemingly different but actually isomorphic, leaving one with the problem of identifying the isomorphic extensions. An extension G may have several normal divisors isomorphic to A. In a fashion natural to the Schreier construction, one of these, A_0, is selected. An isomorphism between two extensions G and H (of A by B) is said to be of the first kind if the A_0 of G is carried onto the A_0 of H. A group of transformations Ω is defined on the set of all the Schreier extensions of A by B, each transformation represented by a triple consisting of a function u on B into A with $u(1_B)=1_A$ and a pair of automorphisms φ and ψ of B and A, respectively, multiplication of triples being defined in a special way. The principal result, a partial solution of the isomorphism identification problem, is that G and H, two extensions of A by B, are connected by an isomorphism of the first kind if, and only if, G and H are in the same transitivity class with respect to Ω, all such isomorphisms being realized by elements of Ω. A sufficient condition on A and B is found for this partial solution to be the complete solution of the isomorphism problem. For a somewhat different point of view, see Baer [Math. Z. 38, 375–416 (1934)].
    F. Haimo (St. Louis, Mo.).

Moisil, Gr. C. Sur la représentation des groupes abéliens infinis. I. Acad. Roum. Bull. Sect. Sci. 23, 358–361 (1942).

Moisil, Gr. C. Sur la représentation des groupes abéliens infinis. II. Acad. Roum. Bull. Sect. Sci. 24, 1–4 (1943).

Moisil, Gr. C. Sur la représentation des groupes abéliens infinis. III. Acad. Roum. Bull. Sect. Sci. 24, 5–7 (1943).

Moisil, Gr. C. Sur la représentation des groupes abéliens infinis. IV. Acad. Roum. Bull. Sect. Sci. 24, 79–84 (1943).

The first two notes treat Abelian groups with all elements of order 2. For every such group G there exists a set M

such that G is isomorphic to a subgroup of the set of all mappings of M into the cyclic group of order 2. There exist commutative rings and Boolean rings having G as a subgroup of their additive groups. A construction is given for a suitable set M using the maximal subgroups of G. In the second note the maximal subgroups are used to introduce a topology into G, in terms of which G is a Fréchet L^*-space, and with respect to which the group operation is continuous. The closure of G with respect to this topology is discussed. The third note is a generalization to the case of Abelian groups with $pa=0$ for all a in G. If p is prime or a product of distinct primes theorems hold similar to those for the case $p=2$. The general case can be reduced to the primary case $p=q^r$, q prime. This case is illustrated for $q=r=2$. The fourth note treats applications of the theory for $p=2$ to the calculus of propositions.   R. M. Thrall.

Černikov, S. N. Infinite groups with finite layers. Mat. Sbornik N.S. **22**(64), 101–133 (1948). (Russian)

In a group the set of all elements of a given order form a layer. A group is said to have finite layers provided the group is periodic and all of its layers are finite sets. A group with finite layers is called thin or thick according as the group does not or does contain an infinite Sylow subgroup. If G is a group with finite layers, every factor group of G has finite layers; moreover, if G is thin, so is every factor group. If the center of a group G has a subgroup N with finite layers and if G/N has finite layers, then G has finite layers; moreover, if N and G/N are thin, so is G. If G is a thick group with finite layers, the center Z of G contains a complete group other than 1; if R is the maximal complete subgroup of Z, then G/R is a thin group with finite layers. An infinite group G is a thin group with finite layers if and only if G is the sum of an ascending chain of finite normal subgroups of G such that, for each prime number p, only finitely many factors of the chain have orders divisible by p. Primary-central extensions of a group are considered with reference to existence and uniqueness; a primary-central extension is so called because, in brief, it involves enlarging certain direct factors of a subgroup of the center to primary Abelian groups of type p^∞. A group is a thick group with finite layers if and only if the group is a primary-central extension of a thin group with finite layers. Call the following condition (*): the center Z of a group G contains a thin subgroup R with finite layers such that G/R is a direct product of a thin group with finite layers and a complete Abelian group A with finite layers; now if G is a thick group with finite layers, (*) holds; if (*) holds, then G has finite layers and moreover the maximal complete subgroup of Z is isomorphic to A. Every complete Sylow subgroup of a group with finite layers is a direct factor of the group. An infinite thin group G with finite layers contains a characteristic subgroup K such that G/K is a direct product of infinitely many finite groups with relatively prime orders. Results are also obtained for a group G with finite layers such that the set of primes dividing the orders of the elements of G is infinite; e.g., such a group contains a characteristic subgroup decomposable into an infinite direct product of finite groups. If G is a group with finite layers, if $R_1=1$, and if, for each integer i, R_{i+1}/R_i is a product of minimal invariant subgroups of G/R_i, then the set-theoretic sum $\sum R_i = G$. In case there exists no non-Abelian simple group of odd order, then every group with finite layers is an extension of a locally solvable group with finite layers by a finite group.   R. A. Good (College Park, Md.).

Černikov, S. N. On the theory of complete groups. Mat. Sbornik N.S. **22**(64), 319–348 (1948). (Russian)

This paper on complete groups with ascending central series continues an earlier study [Rec. Math. [Mat. Sbornik] N.S. **18**(60), 397–422 (1946); these Rev. **8**, 311]. Throughout this review, G denotes an arbitrary complete group with at least two elements and an ascending central series, N denotes an arbitrary normal subgroup of G and R is the additive group of rational numbers. If $G/N\cong R$, then G contains a subgroup A such that $A\cong G/N$, $N\cap A=1$ and $NA=G$. If N is complete and G/N is periodic, then N is a direct factor of G. If G is without torsion and N is complete, then G/N is without torsion. If G/N is without torsion, then N is complete. If G is without torsion, the intersection of every complete subgroup with every complete normal subgroup is a complete group. If G contains elements of infinite order, then G possesses a descending chain of complete characteristic subgroups $G=G_0\supset G_1\supset\cdots\supset G_\alpha\supset\cdots\supset G_\gamma\supset G_{\gamma+1}=1$, where $\gamma\geq 1$, G_β is the intersection of preceding terms whenever β is a limiting ordinal, $G_\alpha/G_{\alpha+1}$ is a direct product of groups isomorphic to R for all $\alpha<\gamma$, and G_γ is the maximal periodic subgroup of G. A group with at least two elements and an ascending central series is complete if and only if it possesses an ascending radical series. Every complete subgroup of G is a term in at least one ascending radical series of G. Every complete N is a term in at least one principal radical series of G; moreover, if N is complete, proper, and without torsion, then N is a term in a principal radical series of G such that all the factors formed from terms contained in N are isomorphic to R. If the maximal periodic subgroup of G has a direct decomposition Δ with indecomposable factors, then between the nontrivial factors of Δ and the periodic factors of an arbitrary ascending radical series of G there exists a one-to-one correspondence such that corresponding groups are isomorphic. The sets of factors of two arbitrary ascending radical series of G may be put in one-to-one correspondence such that corresponding groups are isomorphic. The group G possesses a principal radical series of finite length if and only if G satisfies the "minimal condition for complete subgroups": namely, every descending chain of complete subgroups of G possesses only finitely many terms. All terms in the upper central series of G are complete groups and all factors of this series, after the first, are direct products of groups isomorphic to R. The factor group of G modulo its commutator subgroup is periodic if and only if G is periodic and Abelian. The group G admits a certain type of generalized direct product representation in terms of subgroups isomorphic either to R or to primary Abelian groups of type p^∞.   R. A. Good (College Park, Md.).

Montgomery, Deane. Dimensions of factor spaces. Ann. of Math. (2) **49**, 373–378 (1948).

Si G est un groupe localement compact de dimension finie et que H soit un sous-groupe abélien fermé, l'auteur démontre que la dimension de l'espace facteur G/H est fini. [Un théorème de H. Freudenthal [Ann. of Math. (2) **37**, 46–56 (1936)], dit que si dim $G=0$ et H est diviseur normal, alors dim G/H est aussi 0.] L'auteur se sert des lemmes suivants. (1) Si G est localement compact et H est fermé et que H^* soit le composant de l'identité de H, alors dim $G/H\geq\dim G/H^*$. Ce lemme se démontre en choisissant une série de sous-groupes ouverts et fermés de H, dont l'intersection est l'identité. (2) Si G_x est le sous-groupe des éléments permutables avec x, alors dim $G/G_x\leq\dim G$. (3) Si

$H_2 \subset H_1$ sont deux sous-groupes de G, alors $\dim G/H_2 \leq \dim G/H_1 + \dim H_1/H_2$. *H. Freudenthal* (Utrecht).

Chevalley, Claude, and Eilenberg, Samuel. Cohomology theory of Lie groups and Lie algebras. Trans. Amer. Math. Soc. **63**, 85–124 (1948).

Ce mémoire part d'une conception assez abstraite de la homologie dans les groupes de Lie basée sur une notion d'homologie dans le système des formes différentielles qui se sert des définitions usuelles de forme fermée ($d\omega = 0$) et forme exacte ($\omega = d\omega_1$). Les formes considérées sont les fonctions q-linéaires alternantes définies dans les espaces vecteurs d'une variété M et dont les valeurs se trouvent dans un espace linéaire donné V (pour V=corps des nombres réels, on a les formes ordinaires). Si M est l'espace d'action d'un groupe (T_g) et V celui d'une représentation (P_g) de ce groupe, on appelle ω une forme équivariant dès que $P_g\omega = \omega T_g$. La notion de cohomologie peut être fondée sur toutes les formes ou seulement sur les formes équivariantes. On sait que pour les groupes compacts et les formes ordinaires ces notions ne sont pas différentes (ce qu'on démontre au moyen de l'intégration dans les groupes compacts). A part de ce fait les auteurs démontrent que si P est irréductible et non-trivial, le groupe de cohomologie équivariante est toujours trivial.

Au deuxième chapitre les auteurs procèdent à la "localisation" des formes, c'est-à-dire à leur réduction à un seul point de M. Si G est transitif, le postulat de l'équivariance permet de prolonger d'une manière unique chaque forme locale. Si G est un groupe de Lie qui agit dans $M=G$ par multiplication à gauche, et si la "localisation" a lieu à l'identité, on démontre que la différentiation d'une forme ordinaire ω de degré q consiste à remplacer chaque différentielle par un commutateur de deux différentielles nouvelles et à diviser la somme de ces termes par q, c'est-à-dire si $\omega = \sum a_{r_1,\dots,r_q} dx^{r_1} \cdots dx^{r_q}$, on a

$$d\omega = q^{-1}\sum a_{r_1,\dots,r_q}\sum_i dx^{r_1} \cdots (c_{ih}^k dx^h dx^l) \cdots dx^{r_q}.$$

Une formule analogue est valable pour les formes généralisées. Le postulat supplémentaire d'invariance (ou d'équivariance) droite se traduit par

$$\sum_i \sum a_{r_1,\dots,r_q} dx^{r_1} \cdots [dx^k dx^{r_i}] \cdots dx^{r_q} = 0$$

(et par une formule analogue pour les formes généralisées). On peut se borner aux formes invariantes; leur anneau est isomorphe à l'anneau de cohomologie. Les auteurs aboutissent à démontrer des théorèmes analogues pour les espaces homogènes.

Au troisième chapitre ils prononcent toutes les définitions de nouveau pour les algèbres de Lie sur un corps de caractéristique 0, en récapitulant verbalement les formules infinitésimales auxquelles ils étaient parvenus dans l'étude des groupes de Lie. Quant aux algèbres semi-simples ils partent d'une définition qui consiste à demander l'irréducibilité complète de chaque représentation. Quelques théorèmes sont démontrés à l'aide de la restriction unitaire et du retour au point de vue global; d'autres théorèmes sont traités indépendamment avec des méthodes purement algébriques. On retrouve par exemple le théorème de Hopf (pour les groupes et algèbres non-compacts semi-simples).

Au quatrième chapitre les formes généralisées sont l'objet d'étude. Au cas semi-simple on trouve que tous les groupes de cohomologie sont triviaux. Ce théorème, qui est la tra-

duction d'un lemme de J. H. C. Whitehead, est démontré sans recourir au point de vue global. De la même manière on retrouve le critère de semi-simplicité qui dit que le premier groupe de cohomologie doit être trivial pour chaque représentation [Hochschild]. Enfin les extensions d'algèbres de Lie sont traitées et spécialement les extensions centrales, qui sont déterminées par le deuxième groupe de cohomologie (traduction d'un théorème de Ado). *H. Freudenthal.*

Finzi, A. Sulle trasformazioni singolari di un gruppo continuo e finito e sulle trasformazioni, che non posseggono parametri canonici. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 204–210 (1947).

Soit G_r un groupe de Lie à r paramètres. On sait qu'une transformation de G_r est dite singulière si la substitution correspondante du groupe adjoint admet une racine de Killing égale à $2k\pi i$ ($k \neq 0$). L'auteur démontre les résultats suivants. (1) Si G_r ne possède aucune transformation singulière, on peut attribuer des paramètres canoniques bien déterminés à toute transformation du groupe. (2) Les transformations singulières forment une variété dont la dimension est au plus $r-2$ dans le cas complexe, $r-3$ dans le cas réel. L'auteur illustre enfin ces résultats en examinant le cas du groupe réel unimodulaire à deux variables, pour lequel il retrouve des résultats bien connus et élémentaires. Les méthodes utilisées par l'auteur sont, en général, celles de S. Lie, c'est dire qu'elles ne sont pas toujours parfaitement satisfaisantes. *R. Godement* (Nancy).

Lee, H. C. Sur les groupes de Lie réels à trois paramètres. J. Math. Pures Appl. (9) **26** (1947), 251–267 (1948).

The author classifies all real three-dimensional Lie algebras and represents each of them as the algebra of a suitable linear group. However, three-dimensional Lie groups are not classified from the global point of view.

C. Chevalley (Princeton, N. J.).

Rees, D. On the ideal structure of a semi-group satisfying a cancellation law. Quart. J. Math., Oxford Ser. **19**, 101–108 (1948).

This paper deals with semigroups S having an identity element and satisfying the left cancellation law: $ab = ac$ implies $b = c$. Let $P(S)$ be the set of all principal right ideals $(a) = aS$ of S, partially ordered by inclusion. In general, a partially ordered set is called uniform if it is order-isomorphic with every section P_a of itself (set of all $x \leq a$ in P). As seen from the mapping $(x) \to (ax)$, $P(S)$ is uniform. Conversely, if P is a uniform partially ordered set, the set $S(P)$ of all order-isomorphisms of P with sections of itself is a semigroup under iteration, and $P(S(P)) \simeq P$. A subgroup N of the group G of units of S is called a right normal divisor of S if $Nx \subseteq xN$ for every x in S. Since $aN \cdot bN = abN$, S is homomorphic with the semigroup S/N of cosets, and $P(S/N) \simeq P(S)$. The set M of all g in G such that $gx \in xG$ for every x in S is a right normal divisor of S containing all others. If $P(S) \simeq P$, then $S/M \simeq S'$, where S' is the subsemigroup of $S(P)$ consisting of those mappings of P analogous to the mappings $(x) \to (ax)$ of $P(S)$. Every principal right ideal of $S(P)$ has at least one generator in S', and any two such generate the same principal right ideal in S'. These results are then applied to the determination of all S for which $P(S) \simeq$ the set of integers ordered by magnitude.

A. H. Clifford (Baltimore, Md.).

Rees, D. On the group of a set of partial transformations. J. London Math. Soc. **22** (1947), 281–284 (1948).

By a partial transformation of a set S is meant a one-to-one mapping of one subset of S onto another. A set Σ of partial transformations of S is called regular if (1) α, $\beta\epsilon\Sigma$ imply that the product $\alpha\beta$ is defined and in Σ, and (2) $\alpha\epsilon\Sigma$ implies the inverse mapping $\alpha^{-1}\epsilon\Sigma$. Define $\alpha\sim\beta$ if α and β are extensions of some mapping γ in Σ. This is a congruence relation with respect to which Σ is a group G. Let S be a semigroup satisfying both cancellation laws, and let Σ be the set of all partial transformations of S expressible as a finite product of the partial transformations $x\rightarrow xa$ and their inverses. Then Σ is regular if and only if to each pair a, $b\epsilon S$ there exist x, $y\epsilon S$ such that $xa=yb$. In this case S is isomorphic with a subsemigroup of the group G of congruence classes of Σ. This yields a simple proof of Ore's imbedding theorem [Ann. Math. (2) **32**, 463–477 (1931)] applied to semigroups rather than to noncommutative domains of integrity. *A. H. Clifford* (Baltimore, Md.).

NUMBER THEORY

Carrese, Pietro. Alcune proprietà delle terne pitagoriche intere. Matematiche, Catania 1, 163–170 (1946).

Carrese, Pietro. Qualche altra proprietà delle terne pitagoriche intere. Matematiche, Catania 2, 80–83 (1947).

Becker, H. W., and Riordan, John. The arithmetic of Bell and Stirling numbers. Amer. J. Math. **70**, 385–394 (1948).

The generalized Stirling numbers $S(c, r, s)$ may be defined from $x(x-1) \cdots (x-r-1) = \sum_i S(i, r, -1)x^i$, and $S(c, r, 0) = \delta(c, r)$ (Kronecker delta) and the matrix multiplication rule $S(c, r, s) = \sum_i S(c, i, s-t)S(i, r, t)$. The Bell numbers $B(r, s)$ are defined by the rule $B(r, s) = \sum_c S(c, r, s)$. For the Bell numbers $B(r, 1)$ the period $(p^p-1)/(p-1)$ modulo a prime p was found by the reviewer in an unpublished article. Bell, Touchard, and recently Williams have studied these numbers. Besides a brief summary of these known results, this paper establishes a variety of congruences modulo p on the generalized Bell and Stirling numbers. If $p^{m-1} \leq s < p^m = q$, then $B(r, s)$ has the period p^q-1 modulo p. It is noted that $B(r, 2)$ has the full period 26 modulo 3 and not 13, the period of $B(r, 1)$. The period of the generalized Stirling numbers is also found. There are also a number of congruences involving operators. *M. Hall, Jr.* (Columbus, Ohio).

Ljunggren, Wilhelm. Sur un théorème de M. E. Jacobsthal. Avh. Norske Vid. Akad. Oslo. I. 1947, no. 5, 14 pp. (1948).

For $p|n$, p prime, put $S_\lambda(n, p) = \sum_{\sigma=1}^n \sigma^{-\lambda}$, where the summation is restricted to p not dividing σ; let $\beta_{2s} = (-1)^{s-1}B_s$ in the ordinary notation for Bernoulli numbers. Also let $a_\lambda^{(c)} = \sum \sigma^{-\lambda}$, where $1 \leq \sigma \leq \frac{1}{2}(p^c-1)$; $n = p^c n_1$, p does not divide n_1. A typical result of the paper is the following:

$$S_\lambda(n, p) = hna_{\lambda+1}^{(c)} + 2\frac{n}{p^c}\sum_s \binom{2s+h-1}{h-1} \beta_{2s}p^{2sc}a_{2s+\lambda}^{(c)} \pmod{p^{3r-1}},$$

where $0 \leq s \leq (2r+c-1)/2c$, $p>3$, h even. As an application it is shown that for $p>3$

$$\binom{p^{r+1}}{p^r} - \binom{p^r}{p^{r-1}} = (-1)^{\frac{1}{2}(p-1)}B_{\frac{1}{2}(p-3)}p^{3r+2}/3 \pmod{p^{3r+3}},$$

which is a refinement of a result of Jacobsthal. *L. Carlitz* (Durham, N. C.).

Klee, V. L., Jr. A generalization of Euler's ϕ-function. Amer. Math. Monthly **55**, 358–359 (1948).

We call an integer kth-power-free if it is not divisible by the kth power of any integer greater than 1. If k and n are positive integers we denote by $\Phi_k(n)$ the number of integers h in the set 1, $\cdots$, n, for which the greatest common divisor (h, n) is kth-power-free. Φ_1 is the ϕ-function. *Extract from the paper.*

Munteanu, Octav. Applications du problème de prolongement d'un système associatif dénombrable. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 19–21 (1947).

The following problem is considered. With what freedom can $f(x, y)$ be defined such that (1) f is an integer, (2) $f(m, n) = m+n$ for m and n integers, (3) the associative law $f(x, f(y, z)) = f(f(x, y), z)$ holds? It is found that f is completely defined by the values of $f(x, 0)$, which can be assigned arbitrarily. Furthermore, the analogous result is obtained when hypothesis (2) is changed to $f(m, n) = mn$ for m and n integers. That is, $f(x, y)$ is uniquely determined by the values of $f(x, 1)$, which can be assigned arbitrarily. The author's conditions $x \geq 0$, $y \geq 0$ seem to the reviewer to be not pertinent. *H. Campaigne* (Arlington, Va.).

Rosenthall, E. On the sum of cubes. Bull. Amer. Math. Soc. **54**, 366–370 (1948).

A method is given for obtaining the complete rational integer solution for the Diophantine equation $\sum_{i=1}^m z_i^3 = 0$ $(m>3)$. The solution is of the form $z_i = tu_i/d$, $u_i = P_i(p_1, \cdots, p_M)$, where t, p_1, $\cdots$, p_M are integral parameters, P_1, $\cdots$, P_m are polynomials with integral coefficients and d is the greatest common divisor of the numbers u_1, $\cdots$, u_m. *N. G. de Bruijn* (Delft).

Kahanoff, Boris. Sur le théorème de Fermat. Bull. Inst. Égypte 28, 11–20 (1947).

The author attempts to prove that in case integers x, y, z exist satisfying the Fermat equation $x^n+y^n=z^n$, then the smallest one of them must contain at least 27 digits, no matter what the value of n may be. The proof is not valid, however. The paper also contains some unconvincing remarks about the impossibility of the Fermat equation in case n is nonintegral but rational. *H. W. Brinkmann.*

Aude, Herman T. R. The pattern for the distribution of the numbers c when the Diophantine equation $ax+by=c$ has exactly n solutions. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 10–18 (1947).

Let $f(c)$ denote the number of positive integral solutions of the equation indicated in the title, where a, b, c are positive integers with $(a, b) = 1$. It is proved that $f(c+ab) = 1+f(c)$, $f(c) = 0$ for $1 \leq c \leq a+b-1$, and $f(c) = 1-f(ab+a+b-c)$ for $a+b \leq c \leq ab$. The proofs are based on the classical result that the least and greatest values of c for which $f(c) = n$ are $(n-1)ab+a+b$ and $(n+1)ab$, respectively. *I. Niven.*

Gloden, A. Sur la résolution de la congruence $X^4+1\equiv0$ (mod.p). Euclides, Madrid **8**, 4–5 (1948).

The solutions of $x^4+1\equiv0$ (mod p), which exist if and only if $p=8k+1$, can be written in terms of the solutions of $y^2+1\equiv0$, $2u^2+1\equiv0$, $2v^2-1\equiv0$ (mod p). The author gives $x=\pm(u\pm v)$ as well as the known representation $x=\pm(y\pm1)u$. *I. Niven* (Eugene, Ore.).

Venkataraman, C. S. A theorem on residues and its bearing on multiplicative functions with a modulus. Math. Student **14** (1946), 59–62 (1948).

The author remarks that the Ramanujan sum

$$C_M(N)=\sum_h{}' \exp\,(2\pi i h/M),$$

where the sum is extended over the values of h for which $1\leq h\leq M$, $(h, M)=(N, M)$, is multiplicative in both arguments, that is to say $C_M(N)C_{M'}(N')=C_{MM'}(NN')$ if $(MN, M'N')=1$ This remains true if the exponential function is replaced by any function $f(M, h)$ which satisfies $f(M, h)f(M', h')=f(MM', hM'+h'M)$ whenever M is prime to M'. *N. G. de Bruijn* (Delft).

Negoescu, Nicolae. Sur des approximations asymétriques. C. R. Acad. Sci. Paris **226**, 1495–1497 (1948).

Le problème de la meilleure approximation asymétrique d'un nombre irrationnel θ par des rationnels consiste de trouver la valeur maximum de ξ telle qu'il existe une infinité de fractions p/q qui satisfassent à

$$(1)\qquad -1/(\xi q^2)<p/q-\theta<\tau/(\xi q^2),\qquad \tau>0.$$

La borne supérieure de l'ensemble (ξ) des nombres ξ pour lesquels (1) soit satisfaite par une infinité de fractions rationnelles p/q soit désignée par $M(\theta, \tau)$. L'auteur donne quelques considérations sur ce nombre $M(\theta, \tau)$ à l'aide de la théorie des fractions continues et en donne une représentation géométrique. *J. F. Koksma* (Amsterdam).

Hinčin, A. Ya. A quantitative formulation of the approximation theory of Kronecker. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 113–122 (1948). (Russian)

Let θ_{ij} and α_j $(1\leq i\leq m, 1\leq j\leq n)$ be real constants, and

$$S_j=\sum_{i=1}^m\theta_{ij}x_i-y_j,\qquad T_i=\sum_{j=1}^n\theta_{ij}a_j+a_{n+i},$$

where x_i $(1\leq i\leq m)$, y_j $(1\leq j\leq n)$, a_k $(1\leq k\leq m+n)$ are sets of integer variables. Let A_a be the (nonnegative) distance of $\sum_{j=1}^n\alpha_j a_j$ from the nearest integer, let $a=\max_{j=1}^n|a_j|$ and $T_a=\max_{i=1}^m|T_i|$. Let $\phi(t)$ be any positive continuous and nondecreasing function of the real variable t, and let c_1 and c_2 be constants. The author is concerned with finding conditions under which the inequalities (1) $|S_j-\alpha_j|<c_1 t^{-1}$ $(1\leq j\leq n)$ and (2) $|x_i|<c_2\phi(t)$ $(1\leq i\leq m)$ are soluble in integers x_i, y_j for every $t>0$. The classical theorem of Kronecker [see Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, chap. 7] states a necessary and sufficient condition for (1) alone to be soluble, viz. that $A_a=0$ for every set of integers a_k for which $T_a=0$. In this paper a necessary and sufficient condition is found for both (1) and (2) to be soluble for some c_1, c_2 and every $t>0$, viz. that (i) $A_a=0$ for every set of integers a_k for which $aT_a=0$ and (ii) there exists a positive constant Γ such that $A_a\leq\Gamma a/(\psi(a/T_a))$ for all other sets a_k, where $\psi(t)$ is the inverse function to $t\phi(t)$.

Kronecker's theorem is the limiting case $\phi(t)\to\infty$ of the main theorem. There is another limiting case of interest,

when $\phi(t)$ is constant, $\psi(t)=t$; this case gives the following new theorem. For the equations $S_j=\alpha_j$ $(1\leq j\leq n)$ to be soluble in integers, it is necessary and sufficient that $A_a\leq\Gamma T_a$ for some constant Γ and every set of integers a_k. This theorem was previously proved by the author only in the case $m=1$ [Doklady Akad. Nauk SSSR (N.S.) **56**, 563–565 (1947); these Rev. **9**, 227].

The proof of the main theorem starts from the identity

$$(3)\qquad \sum_{j=1}^n a_j(S_j-\alpha_j)=\sum_{i=1}^m x_i T_i-\sum_{j=1}^n a_j\alpha_j-g,$$

where $g=\sum_{i=1}^m a_{n+i}x_i+\sum_{j=1}^n a_j y_j$ is an integer. From this the "necessity" half follows trivially. The proof of sufficiency is still elementary, but too long for adequate summary here. The idea is to construct $m+n$ independent sets of integers a_{lk} $(1\leq l\leq m+n, 1\leq k\leq m+n)$ such that $|a_{lj}|\leq\lambda^n K_l$, $|T_i(a_{lk})|\leq\lambda^{-n}K_l$ $(1\leq j\leq n, 1\leq i\leq m, 1\leq l\leq m+n)$ with $\lambda=(t\phi(t))^{1/(m+n)}$ and $K_1K_2\cdots K_{m+n}\leq(m+n)!$. That this is possible was proved by Mahler [Nederl. Akad. Wetensch., Proc. **41**, 634–637 (1938)]. These sets a_{lk} then give $m+n$ equations of the form (3); the right-hand sides can be made simultaneously small by a single choice of integers x_i, y_j which makes each "g" equal minus the nearest integer to the corresponding $\sum_{j=1}^n a_j\alpha_j$; the equations being solved for the variables $S_j-\alpha_j$ can then be shown to lead to the inequalities (1). The equations defining the x_i, being also solved explicitly, show that the same choice of x_i satisfies (2). *F. J. Dyson* (Princeton, N. J.).

Godunov, S. K. On a problem of Minkowski. Doklady Akad. Nauk SSSR (N.S.) **59**, 1525–1528 (1948). (Russian)

Let u, v be two homogeneous linear forms with unit determinant in x, y; let λ be the minimum of $|uv|$ for integral nonzero x, y. The author proves that for any real numbers α, β there exist integral x, y such that

$$|(u+\alpha)(v+\beta)|\leq\tfrac14(1-4\lambda^2)^{\frac12},$$

and exhibits various forms for which the constant either is, or is not, best possible. The proof is elementary, and modelled on the proof by Khintchine [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **10**, 281–294 (1946); these Rev. **8**, 444] of the corresponding theorem for the minimum of $|x(\theta x-y-\alpha)|$. Standard methods of the geometry of numbers are used; after a preliminary transformation to bring one generating vector of the (u, v) lattice to an angle of 45° with the axes, the proof utilises only the shape of the region $|uv|\leq C$ in the neighborhood of the origin. *F. J. Dyson* (Princeton, N. J.).

Perron, Oskar. Ein Analogon zu einem Satz von Minkowski. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1945/46, 159–165 (1947).

The author makes the following conjecture. Let α, β, γ, δ, ρ, σ be complex numbers such that $\alpha\delta-\beta\gamma=1$; then there exist integers x, y in the field $K(i\sqrt{D})$ ($D>0$ and "quadratfrei") such that

$$|\alpha x+\beta y-\rho|\cdot|\gamma x+\delta y-\sigma|\leq\begin{cases}\tfrac14(1+D), & D\not\equiv3\pmod4,\\ \tfrac1{16}(1+D)^2/D, & D\equiv3\pmod4;\end{cases}$$

he proves that the right hand member cannot be replaced by a smaller one, states that he possesses a proof of the conjecture for $D=1, 2, 3$ and gives the proof for $D=1$. *J. F. Koksma* (Amsterdam).

Ammann, André. Sur une application d'un théorème de calcul intégral à l'étude des répartitions module 1. C. R. Séances Soc. Phys. Hist. Nat. Genève **64**, 58–61 (1947).

L'auteur désigne d'une façon générale par δ un intervalle (fermé à gauche) pris sur le segment $(0, 1)$, par c la longueur de δ, par $\bar{y}$ le reste, module 1, d'un nombre quelconque y, par $\pi^\delta(y)$ la fonction de période 1 qui vaut 1 si $\bar{y}$ est sur δ et 0 dans le cas contraire. Soit une suite de nombres $\{y_n\}$: l'auteur la dit "unifiante" si, en posant $f_n{}^\delta = \sum_{i=1}^n \pi^\delta(y_i)/n$, la suite $\{f_n{}^\delta\}$ admet le point d'accumulation c, et cela pour tout δ; une suite peut être unifiante sans être équirépartie. L'auteur a pu montrer, étant donnée une suite de fonctions $\{y_n(x)\}$, que la suite de nombres $y_n(x)$, où x a une valeur fixée quelconque, est pour presque-tout x une suite unifiante, pourvu que les fonctions $y_n(x)$ satisfassent à des conditions convenables: par exemple, si les $y_n(x)$ sont linéaires de la forme: $y_n(x) = t_n x$, il suffit que les t_n tendent vers l'infini; l'auteur indique des conditions suffisantes assez larges, applicables à les $y_n(x)$ non nécessairement linéaires.

R. Fortet (Caen).

Kendall, David G. On the number of lattice points inside a random oval. Quart. J. Math., Oxford Ser. **19**, 1–26 (1948).

Let $A(x; \alpha, \beta)$ be the number of lattice points inside or on the circle $(u-\alpha)^2 + (v-\beta)^2 = x$. Let

$$\sigma^2(x) = \int_0^1 \int_0^1 (A(x; \alpha, \beta) - \pi x)^2 d\alpha d\beta.$$

(1) The author proves, using the expansion of $A(x; \alpha, \beta)$ in a double Fourier series, that $\sigma(x) = O(x^{\frac14})$, $\sigma(x) = \Omega(x^{\frac14})$ and $\lim_{u\to\infty} u^{-1}\int_1^u x^{-\frac12}\sigma^2(x)dx = a^2$, where $a = 0.676497 \cdots$. (2) Let $\lambda(x)$ be a positive function which increases to $+\infty$ as x tends to $+\infty$ and let the sequence $\{x_r\}$ increase so rapidly that $\sum 1/\lambda^3(x_r) < \infty$. The author proves that, for almost all (α, β) (in the sense of Lebesgue measure),

$$A(x; \alpha, \beta) - \pi x = O(x^{\frac14}\lambda(x))$$

when x tends to infinity through the sequence $\{x_r\}$. (3) The author then replaces the lattice points (μ, ν) by small equal circular spots, say $(u-\mu)^2 + (v-\nu)^2 \le \delta$, where δ is small and positive, and proves that $B_\delta(x) - \pi x = O(x^{\frac14})$ as $x \to +\infty$ for all fixed $\delta > 0$. Here $\pi \delta B_\delta(x)$ denotes the total area of the lattice spots included within the circle $u^2 + v^2 \le x$.

(4) If x is not an integer, then

$$A(x) = A(x; 0, 0) = B_\delta(x) = \pi x + x^{\frac12}\sum_{l=1}^\infty j_\nu(l\delta)r(l)l^{-1}J_1(2\pi(xl)^{\frac12})$$

for all $\nu > \frac12$ and for all sufficiently small δ. Here $r(l)$ denotes the number of representations of the integer l as the sum of two squares, $j_\nu(u) = \pi^{-\nu}\Gamma(1+\nu)u^{-\frac12\nu}J_\nu(2\pi u^{\frac12})$ and J_ν denote the Bessel function. (5) The author says that a series $\sum a_n$ is summable (j_ν) to the sum S if $\lim_{\delta\to 0+} \sum_{n=1}^\infty j_\nu(n\delta)a_n = S$ and proves that summability (j_ν) is regular for $\nu > \frac12$, so that, if $\sum a_n$ is convergent in the ordinary sense to the sum S, then it is summable (j_ν) to S. Thus Hardy's series $\pi x + x^{\frac12}\sum_{l=1}^\infty l^{-\frac12}r(l)J_1(2\pi(xl)^{\frac12})$ is summable (j_ν) to $A(x)$ for all $\nu > \frac12$ if x is not an integer.

(6) The author extends results (1) and (2) to the more general problem in which the circle is replaced by any sufficiently smooth oval curve, free from singularities and points of zero curvature and gives a quantitative assessment of the accuracy of graphical integration, when the area to be measured is bounded by an oval of the type considered.

V. Knichal (Prague).

Tietze, Heinrich. Über real—statt formal—festgelegte Kettenalgorithmen zur simultanen Approximation mehrgliedriger reeller Zahlenverhältnisse. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 1–43 (1947).

The author discusses a generalization, attributed to Poincaré, of the ordinary continued fraction process. Given n positive real numbers $a_1, \cdots, a_n$, of which, say, a_n is the least, one may replace the numbers by $a_1 - a_n, \cdots, a_{n-1} - a_n, a_n$. The operation may be repeated indefinitely, subtracting at each stage the least number, as long as no two of the set are equal. A modified process is that of forming the new set out of a_n and the remainders when $a_1, \cdots, a_{n-1}$ are divided by a_n. The exposition is diffuse, and no important new results are obtained. *H. Davenport* (London).

Tietze, Heinrich. Ein Algorithmus von Poincaré und andere Algorithmen zur Approximation mehrgliedriger reeller Zahlenverhältnisse. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1945/46**, 185–219 (1947).

A number of examples of generalized continued fractions of the kind discussed in the paper reviewed above are worked out, starting in each case with particular algebraic numbers. These examples show that the properties relating to termination, convergence and periodicity, which are familiar in the case $n = 2$, do not generalize, or only generalize to a very limited extent. *H. Davenport* (London).

Tietze, Heinrich. Über die Stäckelschen Lückenzahlen nebst kleinen Randbemerkungen zur Verteilung der Primzahlen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 21–41 (1944).

The author makes elementary remarks about the sieve of Eratosthenes and about a possible approach to Dirichlet's theorem on primes in an arithmetical progression. In conclusion he gives a table on the distribution of primes belonging to the progressions $26n + 1, 5, 21, 25$. This table is of the same sort as those given in a previous paper [Abh. Bayer. Akad. Wiss. Math.-Nat. Abt. (N.F.) no. 55 (1944); these Rev. **8**, 136]. *D. H. Lehmer* (Berkeley, Calif.).

Tietze, Heinrich. Verallgemeinerung einer Meissner-Stäckelschen Vermutung über die Verteilung der Primzahlen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 69–73 (1944).

The conjecture of Meissner and Stäckel is an extension of the conjecture: "there are infinitely many twin primes," and is as follows. Let $a_1, \cdots, a_n$ be any set of integers which does not contain a complete system of residues with respect to any modulus whatever. Then there is an infinity of integers d such that $d + a_i$ is a prime for $i = 1, 2, \cdots, n$. Moreover, in infinitely many cases these primes are consecutive. The author generalizes this conjecture to primes in arithmetic progression. Many examples of such sets of primes in arithmetic progression are given.

D. H. Lehmer (Berkeley, Calif.).

Tietze, Heinrich. Über den Wettlauf von Restklassen bezüglich ihres Gehalts an Primzahlen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1944**, 75–105 (1944).

The author discusses the problem of the densities of primes in the arithmetic progressions $ax + b_i$ for different choices or classes of b_i, prime to a, especially the cases in which the b's are the various cyclic subgroups of the group of residue classes mod a. By way of illustration six pages [pp. 100–105] of tables are given to a statistical discussion

of the primes less than 1100 belonging to each of the 10 residue classes mod 22. The densities of primes in the 4 classes $22n+1$, $22n-1$, $22n+7$, 13, 17, 19, and $22n+3$, 5, 9, 15 are considered in particular. There is also a discussion of the cases $a=262$, $b=1$, 17, 259. [See Abh. Bayer. Akad. Wiss. Math.-Nat. Abt. (N.F.) no. 55 (1944); these Rev. **8**, 136.]

D. H. Lehmer (Berkeley, Calif.).

Tietze, Heinrich. Ein zweiter Beweis eines Satzes über Partitionen. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1947, 45–46 (1947).

The theorem referred to in the title is the following. Let k be a positive integer. Then the number of partitions of n in which no part is divisible by k is the same as the number of partitions of n in which no part is repeated k or more times. For $k=2$ we have a famous theorem of Euler. The author is unaware that this theorem is due to Glaisher [Messenger of Math. **12**, 158–170 (1883)] whose proof is essentially the same as the one presented here. The reviewer has not seen the author's first proof. However, the theorem follows immediately from the identity

$$(1-x^{km})(1-x^m)^{-1}=1+x^m+x^{2m}+\cdots+x^{(k-1)m}.$$

D. H. Lehmer (Berkeley, Calif.).

Anfert'eva, E. A. On the representation of certain special Dirichlet series by definite integrals. Izvestiya Akad. Nauk SSSR. Ser. Mat. **12**, 79–96 (1948). (Russian)

The author studies the functions $\varphi_\nu(s)=\zeta(s+\nu)\zeta(s-\nu)$, where $\zeta(s)$ is the Riemann zeta function. Generalizations are obtained of results which are well known for $\zeta(s)$ [C. L. Siegel, Quellen u. Studien z. Geschichte der Math. **2**, 45–80 (1932)]. The author proves three formulas involving $\varphi_\nu(s)$. Two of these (which are typical) are as follows. (1) Let

$$a_\nu(n)=\sum_{d|n}d^{2\nu}/n^\nu, \qquad \nu\neq 0,\ -\tfrac{1}{2}<\nu<\tfrac{1}{2},$$

$$\psi(x)=\sum_{n=1}^{\infty}a_\nu(n)K_\nu(2\pi nx),$$

$$\xi(s)=\Gamma(\tfrac{1}{2}(s+\nu))\Gamma(\tfrac{1}{2}(s-\nu))(s^2-\nu^2)[(1-s)^2-\nu^2]\pi^s\varphi_\nu(s),$$

$$w(t)=\xi(\tfrac{1}{2}+it).$$

Then

$$w(t)=\frac{\pi}{\sin\pi\nu}\left[-\frac{\zeta(2\nu)\pi^{-\nu}(1-2\nu)}{\Gamma(1-\nu)}\{(\tfrac{1}{2}+\nu)^2+t^2\}\right.$$
$$+\frac{\zeta(-2\nu)\pi^\nu(1+2\nu)}{\Gamma(1+\nu)}\{(\tfrac{1}{2}-\nu)^2+t^2\}\Big]$$
$$+8\{(\tfrac{1}{2}-\nu)^2+t^2\}\{(\tfrac{1}{2}+\nu)^2+t^2\}$$
$$\times\int_1^{\infty}x^{-\frac{1}{2}}\psi(x)\cos(t\ln x)dx.$$

(2) Let $\epsilon=e^{\pi i/4}$, $\bar\epsilon=e^{-\pi i/4}$,

$$N(2\pi\bar\epsilon x, 2\pi\epsilon n)=2\pi\epsilon nK_{\nu+1}(2\pi\epsilon n)J_\nu(2\pi\bar\epsilon x)$$
$$-2\pi\bar\epsilon xJ_{\nu+1}(2\pi\bar\epsilon x)K_\nu(2\pi\epsilon n)$$

and

$$w(\bar\epsilon, x)=\frac{1}{2\pi i}\left\{-\frac{\Gamma(\nu)\zeta(2\nu)}{\pi^\nu}\bar\epsilon^{1+2\nu}\pi J_{1-\nu}(2\pi\bar\epsilon x)\right.$$
$$-\frac{\Gamma(-\nu)\zeta(-2\nu)}{\pi^{-\nu}}\bar\epsilon^{1-2\nu}\pi J_{1+\nu}(2\pi\bar\epsilon x)$$
$$+2x\bar\epsilon^{2\nu}\sum_{n=1}^{\infty}a_\nu(n)\frac{N(2\pi\bar\epsilon x, 2\pi\epsilon n)}{x^2-n^2}\Big\}.$$

Then

$$\pi^{-s}\Gamma(\tfrac{1}{2}(s+\nu))\Gamma(\tfrac{1}{2}(s-\nu))\varphi_\nu(s)=\Gamma(\tfrac{1}{2}(s+\nu))\Gamma(\tfrac{1}{2}(s-\nu))\pi^{-s}$$

$$\times\int\omega(\bar\epsilon, x)x^{-s}dx+\Gamma(\tfrac{1}{2}(1-s+\nu))\Gamma(\tfrac{1}{2}(1-s-\nu))\pi^{s-1}$$

$$\times\int\omega(\epsilon, x)x^{s-1}dx,$$

where the last two integrals are taken along the path $x=(y-i\xi)e^{i\pi/4}$ (for some fixed ξ, $0<\xi<2^{-\frac{1}{2}}$), with y going from ∞ to $-\infty$. The third formula is even more complicated and we omit it here.

The paper is marred by a tremendous number of misprints which render portions of it almost unreadable. Also, the author seems to have overlooked a previous paper on the same question in which many of the results were given [N. Koschliakov, C. R. (Doklady) Acad. Sci. URSS. **2**, 342–345 (1934)]. *H. N. Shapiro* (New York, N. Y.).

Tutte, W. T. A note to a paper by C. J. Bouwkamp. Nederl. Akad. Wetensch., Proc. **51**, 280–282 = Indagationes Math. **10**, 106–108 (1948).

The partitions of rectangles into non-overlapping squares have been studied by associating to them certain electrical networks [cf., e.g., Bouwkamp, same Proc. **49**, 1176–1188; **50**, 58–71, 72–78 = Indagationes Math. **8**, 724–736 (1946); **9**, 43–56, 57–63 (1947); these Rev. **8**, 398]. In this note, three pairs of squared similar rectangles mentioned by Bouwkamp [loc. cit., p. 72] and the corresponding networks are discussed. The latter belong to a whole family of pairs of networks which are obtained by adding wires to two basic networks. If the similar rectangles are made congruent, then the squares which are represented by corresponding additional wires become equal. *P. Scherk* (Saskatoon, Sask.).

Mirsky, L. The additive properties of integers of a certain class. Duke Math. J. **15**, 513–533 (1948).

Let A be any given class of integers greater than 1 and such that any two integers belonging to A are coprime. A number n will be called A-free if it is not divisible by any a of A. Let $Q(n, A, s)=Q(n)$ denote the number of representations of n (order being relevant) as the sum of s A-free integers. The author obtains an asymptotic formula for $Q(n)$ when $\sum_a 1/a$ is convergent, and gives an upper estimate for $Q(n)$ when $\sum_a 1/a$ is divergent.

W. H. Simons (Vancouver, B. C.).

Willerding, Margaret F. Determination of all classes of positive quaternary quadratic forms which represent all (positive) integers. Bull. Amer. Math. Soc. **54**, 334–337 (1948).

Ramanujan found that there are exactly 54 forms $ax^2+by^2+cz^2+du^2$, where a, b, c, d are positive integers and $1\leq a\leq b\leq c\leq d$, which represent all positive integers for integral values of x, y, z, u [Proc. Cambridge Philos. Soc. **19**, 11–12 (1916)]. A. E. Ross [Amer. J. Math. **68**, 29–46 (1946); these Rev. **7**, 274] gave an upper limit to the determinant of any positive quaternary quadratic form with even cross-product coefficients, which represents all positive integers. The author has shown in her dissertation that there are exactly 178 such classes, and describes the methods of proof in this article. *G. Pall* (Chicago, Ill.).

Moses, Irma. On the representation, in the ring of p-adic integers, of a quadratic form in n variables by one in m variables. Bull. Amer. Math. Soc. **54**, 159–166 (1948).

Let S and T be symmetric, nonsingular integral matrices of respective orders m and n. It is proved that if S represents T in the field of reals and integrally-p-adically for every prime p, then S represents T rationally without essential denominator (that is, the denominators in the rational representation may be taken prime to any preassigned integer). The principal new lemma used is the following. Let a prime p be given. Let S be integral and T in the ring J_p of p-adic integers. Let $B'SB = T$ with B in the field R_p of p-adic numbers, and $C'SC = T$ with C in J_p. Then there exists in J_p a matrix D such that $D'TD = T$, and $|B'SCD - T| \neq 0$.

G. Pall (Chicago, Ill.).

Schwarz, Štefan. On Waring's problem for finite fields. Quart. J. Math., Oxford Ser. **19**, 123–128 (1948).

The main result is the following theorem. Let $GF(p^n)$ be a Galois field. Let k be an integer such that $\delta = (p^n - 1, k) \leq p - 1$. Let $s = g(k)$ be the least value of s for which the equation $\xi = x_1^k + \cdots + x_s^k$ has a solution with $x_i \in GF(p^n)$ for every $\xi \in GF(p^n)$. Then (i) $g(k)$ exists, (ii) $1 \leq g(k) \leq \delta$. This is an improvement of the results of L. Tornheim [Duke Math. J. **4**, 359–362 (1938)] and L. Rédei [Acta Univ. Szeged. Sect. Sci. Math. **11**, 63–70 (1946); these Rev. **8**, 138]. It is also pointed out that there are fields $GF(p^n)$ and integers k ($1 \leq k \leq p - 1$) such that $g(k) = k$. Also $g(k)$ may not exist at all.

L. Carlitz (Durham, N. C.).

ANALYSIS

van der Blij, F. On generalizations of the triangle inequality. Simon Stevin **25**, 231–235 (1947). (Dutch)

In any triangle $a^n \leq (b^n + c^n) \max (1, 2^{n-1} \sin^n A/2)$, for $n \geq 2$. In particular this gives $a^n \leq 2^{n-1}(b^n + c^n)$ for any triangle, and $a^n \leq 2^{\frac{1}{2}n-1}(b^n + c^n)$ if the angle A is not obtuse.

A. W. Goodman (New Brunswick, N. J.).

Jecklin, H. Eine geometrische Anwendung der grundlegenden algebraischen Mittelwerte. Elemente der Math. **3**, 61–63 (1948).

Denote by M_{nk} the kth fundamental algebraic mean of the positive numbers $a_1, \cdots, a_n$; i.e., $M_{nk} = \{s_{nk}/\binom{n}{k}\}^{1/k}$, where s_{nk} is the kth elementary symmetric function of the a's. The known inequalities $M_{n1} \geq M_{n2} \geq \cdots \geq M_{nn}$ (equality holding only if all a's are equal), has the following geometrical meaning. Let B_{nk} be the sum of measures of all k-dimensional boundary-elements of an n-dimensional brick. Among all bricks with given B_{nk} the cube has the greatest B_{nk} for $h > k$, and the least B_{nk} for $h < k$.

B. de Sz. Nagy (Szeged).

Aczél, John. On mean values and operations defined for two variables. Norske Vid. Selsk. Forh., Trondhjem **20**, no. 10, 37–40 (1948).

For functions $m(x, y)$ of two variables only, the author shows that the sufficiency conditions of the Kolmogoroff-Nagumo theorem, insuring the existence of a continuous increasing function $f(z)$ for which

$$(1) \qquad m(x, y) = f^{-1}[\tfrac{1}{2}\{f(x) + f(y)\}],$$

can be weakened by replacing the monotony hypothesis that $m(x, y) < m(x', y)$ for all $x < x'$ by the hypothesis that $x < m(x, y) < y$ for all $x < y$. In addition, an analogue of the Kolmogoroff-Nagumo theorem is given, involving a weaker monotony condition, and with (1) replaced by

$$m(x, y) = f^{-1}[\alpha f(x) + \alpha f(y)]$$

for some real α.   E. F. Beckenbach (Los Angeles, Calif.).

Rosser, J. Barkley. The complete monotonicity of certain functions derived from completely monotone functions. Duke Math. J. **15**, 313–331 (1948).

The following theorems are established on the hypothesis that $F(x)$ is completely monotonic and infinitely differen-

tiable for $x \geq 0$. (A) If we define

$$F_{m,n}(x) = \begin{vmatrix} F^{(m)}(0) & F^{(m+1)}(0) & \cdots & F^{(m+n)}(0) \\ F^{(m+1)}(0) & F^{(m+2)}(0) & \cdots & F^{(m+n+1)}(0) \\ \cdots & \cdots & & \cdots \\ F^{(m)}(x) & F^{(m+1)}(x) & \cdots & F^{(m+n)}(x) \end{vmatrix},$$

then $(-1)^m x^{-n} F_{m,n}(x)$ is completely monotonic for $x \geq 0$. (B) If $G(x) = F(x) - \sum_{i=1}^n \lambda_i e^{-c_i x}$, $c_i \geq 0$, and its first $2n - 1$ derivatives all vanish at $x = 0$, then $x^{-2n} G(x)$ is completely monotonic for $x \geq 0$. There is also a further result similar to (B). The proofs depend on a generalization of the notion of a zero of a function and a corresponding extension of Rolle's theorem.   H. Pollard (Ithaca, N. Y.).

Wintner, A. Arithmetically monotone sequences. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 3–9 (1947).

The analogy between Cauchy and Dirichlet multiplication of series, reflected in the analogy between the matrices with respective elements $(-1)^m \binom{n}{m}$ and $\mu(n/m)$ (where μ is the Möbius function, defined to be zero when m does not divide n), leads the author to call a sequence $\{\lambda_n\}$ completely monotone in the arithmetic sense (D-monotone) if $\nabla_m^{i\to n} \lambda = \sum_{d|k} \mu(d) \lambda_{md} \geq 0$. He shows that $\lambda_n = \int_0^\infty n^{-x} dF(x)$, $dF \geq 0$, F bounded, defines a D-monotone sequence, and raises the question of whether this representation is also necessary for D-monotony.   R. P. Boas, Jr.

Mignosi, Giuseppe. Sulle funzioni crescenti o decrescenti. Matematiche, Catania **1**, 83–87 (1946).

Barrow, David F. Expressing a function of three variables in nomograph form. Duke Math. J. **15**, 433–437 (1948).

All functions concerned being holomorphic, this paper gives a necessary and sufficient condition that $f(x_1, x_2, x_3)$ equal a determinant of the third order whose ith row involves only the single variable x_i. The condition is expressed by the identical vanishing of nine determinantal expressions in the partial derivatives of f. The condition is equivalent to Kellogg's [Z. Math. Phys. **63**, 159–173 (1914)] but is symmetrical and simpler. Explicit formulas for the elements of a determinant equal to an f satisfying the condition are given. In addition to the rational operations, these formulas involve only differentiation and finding the common solution of several quadratic equations in a single unknown. The only irrationality which can be involved is a square

root, whereas Kellogg employs ordinary linear differential equations in defining his elements which are not formulated explicitly.   *J. M. Thomas* (Durham, N. C.).

✶Ritt, Joseph Fels. **Integration in Finite Terms. Liouville's Theory of Elementary Methods.** Columbia University Press, New York, N. Y., 1948. ix+100 pp. $2.75.

This little book is the first treatise to deal with the theory of elementary integrals according to Abel and Liouville, i.e., with the question of characterizing those indefinite integrals which reduce to elementary functions. The question itself is, of course, by no means an elementary one: the precise definition of elementary function requires a considerable background of complex function-theory and Riemann surfaces. The author stresses the need for this background, and he attempts, to a certain extent, to supply it, as the book is not solely intended for persons who possess this background. The main topics treated are as follows. Chapters I and II deal with integrals of elementary functions in the sense of Liouville, and with the nonelementary character of the elliptic integrals and Chebyshev's integral. The results of chapters I and II are generalized in chapter III, where a more general notion of elementary function is introduced, which is due to Ostrowski [Comment. Math. Helv. 18, 283–308 (1946); these Rev. 8, 64]. An initial class J of functions, which is closed with respect to the rational operations and differentiation and which includes at least one function not identically zero, is termed a differential field. The class of functions elementary over J of order not exceeding n is then defined recursively as the smallest class which is closed with respect to the operation of algebraic combination and which includes the logarithm and the exponential of any function of order not exceeding $n-1$. Chapters IV and V deal with some related questions, e.g., the existence of functions of all orders, the nature of the inverse functions, etc. Chapter VI studies the integrability or nonintegrability by quadratures, for differential equations such as Riccati's and Bessel's. Chapters VII and VIII contain generalizations to the case of two variables and their application to the problem of implicit representations of functions of one variable. Throughout the book, all functions considered are analytic, possibly many-valued, and are meromorphic in a domain A of the complex plane. The term algebraic combination of a finite system of such functions f_m is understood to mean a function w which satisfies an algebraic equation $\sum a_k w^k = 0$, where the coefficients a_k are polynomials in the f_m.   *L. C. Young*.

Inglis, Charles. **Mathematics in relation to engineering.** J. Inst. Civil Engrs. 29, 276–289 (1948).

Although delivered to a group of engineers, this lecture can well be recommended as reading matter for mathematicians, teachers of mathematics and others giving instruction in mathematical aspects of engineering and science. The lecturer presents a fresh point of view in the treatment of topics in mathematics, one that adds to the inspiration of students of engineering. He discusses harmonic analysis, Taylor's series, power series solutions of differential equations, Laplace's equation and harmonic functions, step-by-step methods, differential equations of simple vibrating systems and beams, and resonance, as examples of topics that can be presented with a simplicity that adds much to the student's inspiration and his understanding of the subject. The teacher of mathematics can see that our present-

day rigorous treatments of many topics and our careful attention to matters of convergence might well be supplemented by a simple introduction to the subject and its applications.   *R. V. Churchill* (Ann Arbor, Mich.).

Theory of Sets, Theory of Functions of Real Variables

✶Kamke, E. **Mengenlehre.** Walter de Gruyter & Co., Berlin, 1947. 160 pp.

"Zweite, durchgesehene Auflage." The first edition appeared in 1928.

Cuesta, N. **Notes on some works of Sierpiński.** Revista Mat. Hisp.-Amer. (4) 7, 128–131 (1947). (Spanish)

The author provides alternate constructions for three problems of Sierpiński [Fund. Math. 3, 109–122 (1922); Rend. Accad. Sci. Fis. Mat. Napoli (4) 10, 355–356 (1940); Pont. Acad. Sci. Acta 4, 207–208 (1940); these Rev. 8, 17; 2, 256]. These are all based on the "decimals" using ordinals less than or equal to some initial number [compare Maximoff, Ann. of Math. (2) 41, 321–327 (1940); these Rev. 1, 206].   *J. W. Tukey* (Princeton, N. J.).

Sierpiński, W. **Remarque sur l'axiome du choix pour l'espace de fonctions continues.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 216–217 (1947).

The author shows that a rule selecting an element from each nonnull closed subset of the (nonseparable) space of bounded continuous functions $f(x)$ defined for $-\infty < x < \infty$ would determine a rule for well-ordering the real numbers. He then proceeds to set down a rule selecting an element from each nonnull closed subset of a separable complete metric space for which a dense subset has been denumerated.
                    L. M. Graves (Chicago, Ill.).

Dienes, Z. P. **Note sur la comparaison des ensembles mesurables B.** J. Math. Pures Appl. (9) 26 (1947), 227–235 (1948).

Two sets E and F are comparable if it is possible to determine in a denumerable set of operations which one of the five cases, $E < F$, $E > F$, $E = F$, $EF = 0$, $EF > 0$, holds. It is shown that the sets of the classes B_0, B_1, B_2 of Baire are comparable. Let R_α be the class of sets containing B_n for all $n < \alpha$. Two sets E_1 and E_2 of R_α are comparable if it can be determined in a denumerable set of operations that $E = 0$, $E \epsilon R_\alpha$. A fundamental result is as follows. Starting from the definition of B_1 as the limit of sets which are the sums of intervals and complements of intervals, two sets in R_1 are comparable; starting from the definition of B_2 as the limit of sums of open and closed sets, two sets in R_2 are comparable.   *R. L. Jeffery* (Kingston, Ont.).

Maharam, Dorothy. **Set functions and Souslin's hypothesis.** Bull. Amer. Math. Soc. 54, 587–590 (1948).

A class S of elements in a Boolean algebra E is a Souslin system if (1) whenever two elements of S are disjoint, then one contains the other, (2) whenever every two distinct elements of a subclass A of S are disjoint, then A is countable and (3) whenever every two elements of a subclass A of S intersect, then A is countable. The author proves that every Souslin system is countable if and only if on each nonatomic Boolean algebra satisfying the countable chain

condition there exists a real valued function f such that (1) f is monotone nondecreasing, (2) f vanishes at 0 only and (3) every nonzero element contains nonzero subelements on which f is arbitrarily small. The author remarks that the existence of such a function f on E alone is not sufficient to ensure the countability of every Souslin system in E, but the existence of such an f for every nonatomic subalgebra of E is sufficient. *P. R. Halmos* (Chicago, Ill.).

Behrend, F. A. The uniform convergence of sequences of monotonic functions. J. Proc. Roy. Soc. New South Wales **81**, 167–168 (1948).

The author generalizes the following theorem of G. Pólya [Math. Z. **8**, 171–181 (1920)]: if a sequence $f_n(x)$ of monotonic functions converges to a continuous function $f(x)$ in $[a, b]$, then the convergence is uniform. [Reviewer's remark. This theorem had already previously been proved by H. E. Buchanan and T. H. Hildebrandt, Ann. of Math. (2) **9**, 123–126 (1908).] The author now considers the case of a discontinuous function $f(x)$ and proves the following theorem. If $f_n(x)$, $f(x)$ are monotonic in $[a, b]$, then $f_n(x) \to f(x)$ uniformly in $[a, b]$ if and only if (1) $f_n(x) \to f(x)$ for all x of an everywhere dense set E in $[a, b]$ containing the set D of all discontinuities of $f(x)$ and (2) $f_n(x-0) \to f(x-0)$, $f_n(x+0) \to f(x+0)$ for all x in D. *A. Rosenthal.*

Giuliano, Landolino. Sulla formula $\left(\dfrac{dx}{ds}\right)^2 + \left(\dfrac{dy}{ds}\right)^2 = 1$. Boll. Un. Mat. Ital. (3) **2**, 181–188 (1947).

The author gives the following examples of continuous rectifiable curves: (1) a curve for which at one point the derivatives dx/ds, dy/ds are both 0; (2) a curve such that, at one point, $dx/ds = k$, $dy/ds = 0$, where k is an arbitrary real number interior to the interval $(0, \frac{1}{2})$; (3) a curve such that $dx/ds = dy/ds = 0$ at all points of a perfect set, of measure 0 and nowhere dense on the curve; (4) a curve such that $ds/ds = k$, $dy/ds = 0$ (where k is arbitrary but interior to the interval $(0, 1)$) at all the points of a set as above (this example is stated without proof). *T. Viola* (Rome).

✶**Smirnov, V. I. Kurs Vysšeǐ Matematiki. [A Course in Higher Mathematics].** Vol. 5. OGIZ, Moscow-Leningrad, 1947. 584 pp.

[Vol. 4 was published in 1941; cf. these Rev. **6**, 42.] This book appears to be written for the Russian equivalent of first year graduate students in America. Its content overlaps greatly with what is frequently taught under the title "real variables," but the choice of topics is somewhat unusual. The book assumes on the part of the reader very little mathematical knowledge beyond the ability to handle elementary epsilontics. Such topics as elementary set theory, cardinal number theory and the topology of the real line are treated (but only briefly, to the extent that they are needed for later developments).

The author's expository style is pleasantly slow and discursive. His attitude is classical; he almost never strives for the greatest possible generality, even at places where he could do so without any sacrifice of either clarity or brevity. Most of the functions considered are defined in Euclidean spaces (usually of one or two dimensions). The chapter on Hilbert space contains a detailed treatment of the functional calculus and spectral theory; the last chapter is concerned mostly with special function spaces such as L^p and C^p.

The principal headings in the table of contents are as follows. Chapter I: Stieltjes integrals. Chapter II: Set functions and the Lebesgue integral (§ 1: Set functions and the theory of measure, § 2: Measurable functions, § 3: Lebesgue integral). Chapter III: Set functions, absolute continuity and the general concept of integral. Chapter IV: Hilbert space (§ 1: Theory of bounded operators, § 2: The spaces l_2 and L_2, § 3: Unbounded operators). Chapter V: General spaces. *P. R. Halmos* (Chicago, Ill.).

✶**Ríos, Sixto. Conceptos de Integral. [Concepts of Integral].** Instituto "Jorge Juan" de Matematicas, Madrid, 1946. 79 pp.

This is a condensation of the author's lectures [Revista Acad. Ci. Madrid **36**, 10–49, 307–354, 418–481 (1942), also issued in book form; these Rev. **7**, 11].

Dwinas, S. An application of the theory of random sampling to the theory of the integral. Revista Mat. Hisp.-Amer. (4) **7**, 234–238 (1947). (Spanish)

The author applies stratified sampling (of subintervals) to the evaluation of Riemann and Lebesgue integrals. His work was anticipated by Adams and Morse [Bull. Amer. Math. Soc. **45**, 442–447 (1939)], who later extended the subject further [Trans. Amer. Math. Soc. **53**, 363–426 (1943); these Rev. **5**, 2]. *J. W. Tukey* (Princeton, N. J.).

Mambriani, A. Sull'approssimazione dell'integrale di Lebesgue per le funzioni di una variabile. Boll. Un. Mat. Ital. (3) **2**, 173–181 (1947).

In preparation for applications in surface area theory, the author establishes the following theorems on approximations to Lebesgue integrals by Riemann-type sums. (I) Let $f(x)$ be summable in $a \leqq x \leqq b$ and let $\epsilon > 0$ be arbitrarily assigned. Then there exists a set of points $x_1, \cdots, x_m$, such that $a \leqq x_1 < \cdots < x_m \leqq b$, $x_1 - a < \epsilon$, $b - x_m < \epsilon$, $x_{i+1} - x_i < \epsilon$, and such that the summation $\sum (x_{i+1} - x_i) f(x_i^*)$, $i = 1, \cdots, m$, differs by less than ϵ from the integral of $f(x)$ from a to b, where x_i^* coincides, for each i, with either x_i or x_{i+1}. (II) Let $f(x)$ be summable in $a \leqq x \leqq b$, and let $\epsilon > 0$ be arbitrarily assigned. Then there exists a $\delta = \delta(\epsilon) > 0$, such that the following holds. Let D be any subdivision of the interval (a, b) into subintervals $l_1, \cdots, l_m$ of lengths less than δ, and let μ_i be the integral mean of $f(x)$ on l_i. Then the sum of the integrals of the expressions $|f(x) - \mu_i|$, taken over l_i, $i = 1, \cdots, m$, is less than ϵ. *T. Radó* (Columbus, Ohio).

Ammann, André. Un théorème concernant les suites infinies de fonctions qui deviennent nulles en moyenne sur tout intervalle. C. R. Séances Soc. Phys. Hist. Nat. Genève **64**, 40–42 (1947).

L'auteur dit qu'une suite infinie $\{f_n(x)\}$ de fonctions réelles devient nulle en moyenne sur un intervalle (a, b) si $\lim_{n \to \infty} \int_a^b f_n(x) dx = 0$; soit une suite $\{f_n(x)\}$ telle que: (α) les $f_n(x)$ sont réelles, mesurables et bornées dans leur ensemble; (β) $\lim_{n \to \infty} [f_{n+1}(x) - f_n(x)] = 0$ pour tout x de (a, b); (γ) $\{f_n(x)\}$ devient nulle en moyenne sur tout intervalle (a', b') intérieur à (a, b); l'auteur montre qu'alors la suite des nombres $f_n(x)$, où x est une valeur fixe quelconque, admet 0 pour point d'accumulation pour presque-tous les x de (a, b). *R. Fortet* (Caen).

Halmos, Paul R. The range of a vector measure. Bull. Amer. Math. Soc. **54**, 416–421 (1948).

Let X be a set and let S be a σ-field of subsets of X. A measure $\mu = (\mu_1, \cdots, \mu_N)$ is a bounded totally additive

function of the sets of S. Let, furthermore, μ be nonnegative, i.e., $\mu_i(E) \geqq 0$ for every $E \varepsilon S$ and $i = 1, \cdots, N$. A set $E \varepsilon S$ is called an atom if $\mu(E) \neq 0$ and if for every $F \subset E$ either $\mu(F) = 0$ or $\mu(F) = \mu(E)$; μ is called purely nonatomic if none of its coordinates μ_i has any atoms. For any measure μ and $E \varepsilon S$ let $K(\mu, E)$ be the class of all real-valued measurable functions ϕ on E for which $0 \leqq \phi(x) < 1$ and $\mu(\{x: \phi(x) < \lambda\}) = \lambda \mu(E)$ for $0 \leqq \lambda \leqq 1$. A measure μ is convex if for every set $E \varepsilon S$ the class $K(\mu, E)$ is not empty. In several former papers [Liapounoff, Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 465–478 (1940); K. R. Buch, Danske Vid. Selsk. Math.-Fys. Medd. 21, no. 9 (1945); Halmos, same Bull. 53, 138–141 (1947); these Rev. 2, 315; 7, 279; 8, 506] the range of a measure μ has been discussed. In this paper the author presents a simplified proof of Liapounoff's results that the range is closed and, in the nonatomic case, convex. *K. R. Buch.*

Halmos, Paul R. Functions of integrable functions. J. Indian Math. Soc. (N.S.) 11, 81–84 (1947).

A real-valued function f of n real variables is called by the author a $(p \rightarrow q)$ function on an interval (a, b) if and only if for every set $(x_1, \cdots, x_n)$ of n real-valued functions defined on (a, b) and belonging to $L_p(a, b)$ the function $g(t) = f(x_1(t), \cdots, x_n(t))$ belongs to $L_q(a, b)$. Then he proves the following theorems. (1) A necessary and sufficient condition that a Baire function f be a $(p \rightarrow q)$ function on $(0, 1)$ is that there exist a positive constant K such that for all points $(y_1, \cdots, y_n)$ the inequality

$$|f(y_1, \cdots, y_n)|^q \leqq K(1 + |y_1| + \cdots + |y_n|)^p$$

is satisfied. (2) The same theorem holds if $(0, 1)$ is replaced by $(-\infty, +\infty)$, and the *A. Rosenthal* (Lafayette, Ind.), term 1 is omitted from the right-hand side of the preceding inequality

Goffman, Casper. A class of rectangle functions. Duke Math. J. 15, 127–135 (1948).

The author considers functions of the form

$$F(I \times J) = \int_J f(I, y) dy,$$

where I, J are linear intervals and $f(I, y)$ is a nonadditive function of I. It is shown that the strong derivative $F_g'(x, y)$ and the bilateral partial derivative $f'(x, y)$ (defined as $\lim f(I, y)/|I|$ as $|I| \rightarrow 0$ with $I \ni x$) are equal at almost all points where they both exist. Examples show that this theorem becomes untrue if we consider the regular derivative $F'(x, y)$ or the unilateral or approximate partial derivatives of f, but appropriate theorems are obtained in these cases also. *A. J. Ward* (Cambridge, England).

Ringenberg, Lawrence A. The theory of the Burkill integral. Duke Math. J. 15, 239–270 (1948).

The author presents a connected account of the Burkill integral, starting with elementary and well-known results, but passing on to discuss the properties of the upper and lower integrals, and more particularly of their derivatives, when they are unequal. Finally, he shows the connection of these results with the well-known derivability property of the Lebesgue integral. The definition of "écart" [p. 251] appears to the reviewer not to say quite what is intended. *A. J. Ward* (Cambridge, England).

Helsel, R. G., and Radó, T. The Cauchy area of a Fréchet surface. Duke Math. J. 15, 159–167 (1948).

If there is given the surface S in parametric form, let p denote a plane through the origin in (x, y, z)-space. Let ξ, η denote Cartesian coordinates in p and let $K_p(\xi, \eta)$ denote the essential multiplicity function associated with the orthogonal projection of S upon p [T. Radó, same J. 14, 587–608 (1947); these Rev. 9, 231]. Let us define α_p as the integral of $K_p(\xi, \eta)$ if this integral exists, otherwise let us put $\alpha_p = +\infty$. Let P be a point on the unit sphere $U: x^2 + y^2 + z^2 = 1$, and let us put $\alpha(P) = \alpha_p$, where p is the plane through the origin that is perpendicular to the radius of the unit sphere that joins P to the origin. If $\alpha(P)$ is summable over the surface of the unit sphere U, the Cauchy area $C(S)$ is defined to be twice the integral mean value of $\alpha(P)$ taken over U; if $\alpha(P)$ is not summable over U we put $C(S) = +\infty$. Let us call $L(S)$ the Lebesgue area of S. In another paper Radó has already stated that $C(S) < +\infty$ implies $C(S) = L(S)$ [Proc. Nat. Acad. Sci. U. S. A. 31, 102–106 (1945); these Rev. 6, 204]. H. Federer has shown that $C(S) = L(S)$ for every rectifiable surface S [Trans. Amer. Math. Soc. 59, 441–466 (1946); these Rev. 7, 422]. In this paper it is proved that always $C(S) = L(S)$ for every surface S. The simple proof uses several concepts and results of Radó and one of the reviewer. *L. Cesari* (Bologna).

Mickle, E. J., and Radó, T. A new geometrical interpretation of the Lebesgue area of a surface. Duke Math. J. 15, 169–180 (1948).

In connection with a previous paper of J. Favard [C. R. Acad. Sci. Paris 194, 344–346 (1932)] the authors give a new geometrical interpretation of the Lebesgue area of a Fréchet surface. Let us consider the set G of all directed lines of the space E_3. The set G can be metrized by a suitable measure μ. For a set E in E_3, let $N(g, E)$ denote the number of intersections of a directed line g with E. If $N(g, E)$ is a summable function on G with respect to the measure μ then the Favard measure $F(E)$ of E is given by $F(E) = \int_G N(g, E) d\mu$. For a surface S in E_3 in parametric form let $N^*(g, S)$ be the minimum of the number $N(g, S^*)$ for every surface S^* in E_3 such that the Fréchet distance $\|S^*, S\|$ is less than ϵ. The number $N_e(g, S) = \lim_{\epsilon \rightarrow 0+} N^*(g, S)$ is called by the authors the essential number of intersections of the directed line g with the surface S. The authors prove that the Lebesgue area $L(S)$ of the surface S is finite if and only if the function $N_e(g, S)$ is summable in G with respect to the invariant measure μ and that always $L(S) = \int_G N_e(g, S) d\mu$. This result holds for every surface S. *L. Cesari* (Bologna).

Theory of Functions of Complex Variables

Ghizzetti, Aldo. Analisi in Italia nel campo complesso (dal 1939 al 1945). Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 5, 35 pp. (1945).

Bers, Lipman. On rings of analytic functions. Bull. Amer. Math. Soc. 54, 311–315 (1948).

Let D be a domain of the complex z-plane and let $R(D)$ denote the totality of single-valued analytic functions defined in D. A one-to-one directly conformal map $\zeta = \phi(z)$ of D onto a domain Δ induces an isomorphism $f \rightarrow f^*$ between $R(D)$ and $R(\Delta): f(z) = f^*[\phi(z)]$. The author establishes the following converse results. If $R(D)$ is isomorphic to $R(\Delta)$, then there exists a directly or inversely conformal map of D onto Δ. If D and Δ possess boundary points then every isomorphism between $R(D)$ and $R(\Delta)$ is induced by a directly or inversely conformal map of D onto Δ. *E. F. Beckenbach* (Los Angeles, Calif.).

Agmon, Shmuel. Fonctions analytiques dans un angle et propriétés des séries de Taylor. C. R. Acad. Sci. Paris 226, 1497–1499 (1948).

The author announces a very general theorem, too complicated to reproduce here, giving an estimate for a function which is of exponential type in an angle and of prescribed growth on a sequence of points in the angle. A consequence is a similarly general estimate for the partial sums of the power series of a function with coefficients of prescribed growth.   *R. P. Boas, Jr.* (Providence, R. I.).

Agmon, Shmuel. Sur deux théorèmes de Fabry. C. R. Acad. Sci. Paris 226, 1673–1674 (1948).

The author gives some applications of the general theorems stated in his note reviewed above. The principal one is a very general formulation of the principle that a power series, with gaps whose lengths tend to infinity, and with the coefficient at one end of each gap almost as large as any coefficient ever is ("principal sequence" of coefficients), has the circle of convergence as a natural boundary. [For older and weaker formulations of the same principle cf. H. Claus, Math. Z. 49, 161–191 (1943); L. Ilieff, Annuaire [Godišnik] Univ. Sofia. Fac. Phys.-Math. Livre 1. 42, 67–81 (1946); these Rev. 5, 176; 9, 21.]   *R. P. Boas, Jr.*

Agmon, Shmuel. Sur le comportement d'une série de Taylor sur le cercle de convergence. C. R. Acad. Sci. Paris 226, 1875–1876 (1948).

The author outlines the proofs of two theorems. The first states that the circle of convergence of $\sum \eta_n \theta_n J_n z^n$ is a natural boundary if the following conditions are all satisfied: $l_{n+1}/l_n \to 1$ and l_n satisfies some conditions of regularity imposed in the note of the second preceding review; $0 < \delta \leq |\theta_n| \leq 1$, $\theta_{n+1}/\theta_n \to 1$; lim sup $\eta_n > 0$, $\{\eta_n\}$ takes only a finite number of values and is not ultimately periodic. [For a weaker theorem of the same character, also generalizing the classical theorem of Szegö, see Ilieff's paper cited in the preceding review.] The second theorem states that if, in $\sum d_n z^n$, $\{d_{n_k}\}$ is a principal sequence [cf. the preceding review] and $n_{k+1} - n_k \to \infty$, and $d_n{}^* = d_n$ for $n \neq n_k$, $d_n{}^* = -d_n$ for $n = n_k$, then the circle of convergence is a natural boundary for at least one of $\sum d_n z^n$, $\sum d_n{}^* z^n$.

It is well known that if $\sum c_n z^n$ converges in $|z| < 1$ and $c_n \to 0$, then the series converges at every regular point on $|z| = 1$. The author discusses the rapidity of this convergence.   *R. P. Boas, Jr.* (Providence, R. I.).

Agmon, Shmuel. Sur un théorème de M. Mandelbrojt. C. R. Acad. Sci. Paris 226, 1786–1787 (1948).

If a power series has an infinity of gaps of length k, with a bounded distance between consecutive gaps, then it has at least $k+1$ singular points on the circle of convergence. The proof is outlined and a more general theorem is stated.   *R. P. Boas, Jr.* (Providence, R. I.).

Blambert, Maurice. Sur les singularités des fonctions analytiques définies par des développements dirichlétiens. C. R. Acad. Sci. Paris 226, 1666–1668 (1948).

Let $f(s)$ and $g(s)$ be two analytic functions given by (convergent) Dirichlet series expansions $f(s) = \sum a_n e^{-\lambda_n s}$, $g(s) = \sum b_n e^{-\mu_n s}$ in some half-plane $\sigma > \sigma_0$. Let n_0 be the least integer such that $\mu_n > \lambda_n$ for $n \geq n_0$ and put

$$a_n{}^{(k)} = \sum_{\lambda_m < \mu_n} (\mu_n - \lambda_m)^k a_m, \qquad n \geq n_0,$$

$$= 0, \qquad\qquad\qquad 1 \leq n < n_0,$$

The "Hadamard-Mandelbrojt product" of $f(s)$ and $g(s)$ is defined as $\sum a_n{}^{(k)} b_n e^{-\mu_n s}$. The author states a theorem for the Hadamard-Mandelbrojt product which is a generalisation of Borel's theorem on the Hadamard product of two power series with polar singularities [Bull. Soc. Math. France 26, 238–248 (1898)]. The statement of the theorem is too long to be reproduced here. The proof is based on results of Mandelbrojt [Rice Inst. Pamphlet 31, 159–272 (1944); these Rev. 6, 267].   *W. H. J. Fuchs* (Ithaca, N. Y.).

Biernacki, Mieczysław. Sur le moyennes de module des fonctions holomorphes. Ann. Univ. Mariae Curie-Skłodowska. Sect. A. 1, 1–8 (1946). (French. Polish summary)

By using the identity,

$$r \frac{d}{dr} \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta = \lambda \int_{|z|=r} R^\lambda d\varphi,$$

valid for $f(re^{i\theta}) = Re^{i\varphi}$ regular in $0 \leq r < 1$ and λ real and positive, which is a special case of one used by H. Prawitz [Ark. Mat. Astr. Fys. 20A, no. 6 (1927)], the author gives a new proof of the theorem of Littlewood that if $f(z)$ and $F(z)$ are regular in $|z| < 1$ and if $f(z)$ is subordinate to $F(z)$, then for every $r < 1$ and for every $\lambda > 0$,

$$\int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |F(re^{i\theta})|^\lambda d\theta,$$

with equality if and only if $f(z) = F(e^{i\psi} z)$ for some real ψ.

The author also proves the following theorem: Let $f(z)$ and $F(z)$ be regular in $|z| \leq r$ and, for every $\rho > 0$, let the sum of the lengths of the arcs of the circumference $|w| = \rho$ which are covered by the values of $f(z)$ when z describes the circle $|z| \leq r$ (multiplicities being counted) not exceed the corresponding sum formed for $F(z)$. Then

$$\frac{d}{dr} \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \frac{d}{dr} \int_0^{2\pi} |F(re^{i\theta})|^\lambda d\theta,$$

with equality only if the two sums are equal for every ρ. In conclusion, the author establishes a connection between the integral means and the characteristic function of Nevanlinna and gives some applications.   *W. Seidel.*

Nehari, Zeev. On analytic functions possessing certain properties of univalence. Proc. London Math. Soc. (2) 50, 120–136 (1948).

According to the classical distortion theorem for univalent functions, if a function $f(z) = z + a_2 z^2 + \cdots$, regular and univalent in $|z| < 1$, omits a value d, then $|d| \geq \frac{1}{4}$. Analogous results are proved here under considerably weaker assumptions on $f(z)$. Thus, let $f(z) = z + a_2 z^2 + \cdots$ be regular in $|z| < 1$ and let $w = d$ be a point on the boundary of the region D on which $|z| < 1$ is mapped by $w = f(z)$. Then, (a) if $|w| < |d|$ is covered by D in a "schlicht" manner (i.e., for every w' with $|w'| < |d|$ the equation $f(z) = w'$ has exactly one solution in $|z| < 1$), the inequality $|d| \geq e^{-2}$ holds, with equality only if $f(z) = e^{-2} z \exp [2(1 - Kz)/(1 + Kz)]$, $|K| = 1$; (b) if a curve connecting $w = d$ with $w = 0$ is covered by D in a "schlicht" manner (i.e., for every w' on this curve the equation $f(z) = w'$ has exactly one solution in $|z| < 1$), the inequality $|d| \geq \pi^{-2}$ holds, with equality only if $f(z) = K^{-1} \pi^{-2} \sin^2 [\pi(Kz)^{\frac{1}{2}}/(1 + Kz)]$, $|K| = 1$. The chief method used is the principle of subordination together with the explicit determination of the mapping functions for certain types of Riemann surfaces.

The last result is extended to the case of bounded functions. Let $w = f(z) = \alpha z + \cdots$ be a regular function for $|z| < 1$ and satisfying there $|f(z)| \leq 1$; let further d $(0 < d < 1)$ be a point of the boundary of the domain D on which $|z| < 1$ is mapped by $w = f(z)$; and let there finally exist a curve which is covered in a "schlicht" manner by D and connects $w = 0$ with $w = d$. Then, $|\alpha| \leq \tau^2 \vartheta_2^4(0; \tau) \tanh^2 \frac{1}{2}\pi/\tau$, where τ $(\tau > 0)$, is given by the equation $|d| = \vartheta_2^2(0; 2\tau)/\vartheta_3^2(0; 2\tau)$, ϑ_2 and ϑ_3 being the Jacobian ϑ-functions. The bound for $|\alpha|$ is the best possible, being attained for a certain function related to the elliptic functions. A similar result is also obtained for doubly-connected regions. *W. Seidel.*

Boas, R. P., Jr., Buck, R. C., and Erdős, P. The set on which an entire function is small. Amer. J. Math. 70, 400–402 (1948).

Soit $f(z)$ une fonction entière quelconque, $M(r)$ le maximum de son module pour $|z| = r$; $D_R(E_\lambda)$ le quotient par πR^2 de la mesure des points z, $|z| \leq R$, en lesquels $\log |f(z)| \leq (1 - \lambda) \log M(|z|)$; $\bar{D}(E_\lambda)$ et $\underline{D}(E_\lambda)$ les limites supérieure et inférieure de $D_R(E_\lambda)$ lorsque $R \to \infty$. Par une simple application du théorème de Jensen, les auteurs montrent que: Si $\lambda > 1$, on a $\bar{D}(E_\lambda)\lambda \leq 1$; si $\lambda \to \infty$, $\underline{D}(E_\lambda) = o(\lambda^{-1})$. Ils prouvent, en outre, sur un exemple, qu'il existe des fonctions entières pour lesquelles $\bar{D}(E_\lambda)$ est positif, au moins égal à $\delta^2/(1+\delta)^2$ avec $\delta(2+\delta)\lambda^{-1} = 1$, $\delta > 0$. Certaines généralisations sont signalées. *G. Valiron* (Paris).

Wang, Fu Traing. Note on Paley-Wiener's theorem. Duke Math. J. 15, 1–3 (1948).

Die Nullstellenanzahl $n(r)$ im Kreis $|z| \leq r$ einer geraden ganzen Funktion $f(z)$ der Ordnung $\rho \leq 1$ genüge der Bedingung $n(r) = Ar \log r + Br + o(r)$. Damit die Nullstellen alle reell seien ist dann notwendig und hinreichend, dass

$$-4\pi^{-1} \int_0^T x^{-2} \log |f(x)| \, dx = A \log T + B + o(1).$$

Mit $A = 0$ ergibt sich das Resultat von Paley und Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934]. *A. Pfluger* (Zürich).

Bowen, N. A. A function-theory proof of Tauberian theorems on integral functions. Quart. J. Math., Oxford Ser. 19, 90–100 (1948).

The principal theorem in question concerns an entire function $f(z)$ of order ρ, $0 < \rho < 1$, with negative real zeros; $n(r)$ denotes the number of zeros of absolute value not exceeding r; then if $\log f(x) \sim \pi x^\rho \csc \pi \rho$ as $x \to +\infty$, it follows that $n(r) \sim r^\rho$. This was proved by Valiron [Ann. Fac. Sci. Univ. Toulouse (3) 5, 117–257 (1914), pp. 237–243] by a lengthy Tauberian argument; by Titchmarsh [Proc. London Math. Soc. (2) 26, 185–200 (1927)] by showing it equivalent to a Tauberian theorem of Hardy and Littlewood; and by Paley and Wiener [Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloquium Publ., v. 19, New York, 1934, p. 79] by using Wiener's general Tauberian theorem. The author gives a much more elementary proof. The same line of argument is shown to apply to more general situations when the zeros only approach the negative real axis, when proximate orders are used or when the order of $f(z)$ is greater than 1 and not integral; these results were known. Finally, the theorem is extended to cases when the behavior of $f(z)$ is prescribed only along a sufficiently dense discrete set of points. [For a substantially equivalent "function-theory" proof see M. Heins, Ann. of Math. (2) 49, 200–213 (1948), pp. 210–212; these Rev. 9, 341]. *R. P. Boas, Jr.* (Providence, R. I.).

Harvey, A. R. The mean of a function of exponential type. Amer. J. Math. 70, 181–202 (1948).

Es sei $f(z)$ eine ganze Funktion vom Mitteltypus k der Ordnung 1,

$$M^{(p)}(y, f) = \lim \sup_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(x+iy)|^p dx$$

und

$$M_i^{(p)}(f) = \lim \sup_{n \to \infty} \frac{1}{2n+1} \sum_{i=-n}^{n} |f(i)|^p.$$

Es ist bekannt, dass im Fall $k < \pi$ die Beschränktheit von $f(\pm n)$ die Beschränktheit von $f(z)$ auf der reellen Achse nach sich zieht. Entsprechendes gilt für Funktionen f, die auf der reellen Achse zur Klasse L^p gehören [Plancherel und Pólya, Comment. Math. Helv. 10, 110–163 (1937)]. Analoge Resultate erhält Verf. für Funktionen f, deren Mittelwerte $M^{(p)}(y, f)$ bzw. $M_i^{(p)}(f)$ beschränkt sind. Zunächst ist $M^{(p)}(y, f) \leq e^{p k |y|} M^{(p)}(0, f)$, $p > 0$, und $M^{(p)}(f) \leq k^p M^{(p)}(f)$, $p > 1$. Bei $k = 0$ und $M^{(p)}(0, f) < \infty$ $(p > 0)$ ist f notwendig eine Konstante. Aus $M^{(p)}(0, f) < \infty$ folgt $M_i^{(p)}(f) < \infty$ und für $k < \pi$ auch umgekehrt. Die Beweismethode ist im wesentlichen dieselbe wie bei Plancherel und Pólya. Die Resultate lassen sich übertragen auf Funktionen, die in einer Halbebene regulär und vom Exponentialtypus sind, wenn statt der obigen sogenannte einseitige Mittelwerte verwendet werden. *A. Pfluger* (Zürich).

Vermes, P. Note on a two-point boundary problem. Quart. J. Math., Oxford Ser. 19, 109–116 (1948).

The problem is that of finding an entire function $f(z)$ such that $f^{(p_i)}(1) = a_i$, $f^{(q_j)}(0) = b_j$, where $\{p_i\}$ and $\{q_j\}$ are given monotonic sequences of nonnegative integers and $\{a_i\}$ and $\{b_j\}$ are given sequences of complex numbers. The author studies the problem by reducing it to a system of infinitely many equations in the infinitely many unknowns $f^{(k)}(0)$, $k \neq b_j$. He obtains the following sufficient condition: (i) each of $\{p_i\}$, $\{q_j\}$ contains an infinity, and omits an infinity, of positive integers; (ii) $\{a_i\}$ and $\{b_j\}$ are bounded; there is a one-to-one correspondence between $\{q_j\}$ and $\{p_i'\}$ (the prime denotes the complementary sequence) such that $q_j = p_i' - t_j$, $t_j = 0$ or 1; (iv) if $t_j = 1$ for an infinity of j, $\lim \inf p_{i-1}/p_i > 0$. Further results deal with uniqueness of the solution and extensions to cases where more than two points are involved. *R. P. Boas, Jr.* (Providence, R. I.).

Selmer, Ernst S. A simple trisection formula for the elliptic $\wp$-function of Weierstrass in the equianharmonic case. Norske Vid. Selsk. Forh., Trondhjem 19, no. 29, 116–119 (1947).

Let $\wp(z) = \wp(z; g_2, g_3)$ denote the Weierstrassian $\wp$-function with the invariants g_2 and g_3. The author shows that, in the particular case $g_2 = 0$, the algebraic equation of the 9th order to which one is led when trying to express $\wp(z/3)$ in terms of $\wp'(z)$ can be solved by the use of trigonometric functions and radicals. *Z. Nehari* (St. Louis, Mo.).

✶Maass, Hans. Über automorphe Funktionen von mehreren Veränderlichen und die Bestimmung von Dirichletschen Reihen durch Funktionalgleichungen. Ber. Math.-Tagung Tübingen 1946, pp. 100–102 (1947).

This paper outlines a generalization of Hecke's theory of modular functions. Hecke obtains, by means of the Mellin

transformation, Dirichlet series whose functional equations are equivalent to those of the modular functions. The author obtains a wider class of Dirichlet series by considering automorphic functions of several variables. For real variables $x_0, \cdots, x_k$ in the region $x_k > 0$ with the metric $ds^2 = (dx_0^2 \cdots dx_k^2)/x_k^2$ he considers those solutions $g(x)$ $(x = \{x_0, \cdots, x_k\})$ of $\Delta g + (r^2 + \frac{1}{4}k^2)x_k^{-(k+1)}g = 0$ (Δ the Beltrami operator, r a parameter) which show the invariance $g(x+a) = g(x)$ for all $a = \{a_0, x_1, \cdots, a_{k-1}, 0\}$ of a k-dimensional lattice, and also the invariance $g(x/|x|^2) = g(x)$, with $|x|^2 = x_0^2 + \cdots + x_k^2$. A Fourier expansion leads to functional equations $F(1/x_k, P_n) = F(x_k, P_n)$, $n = 0, 1, 2, \cdots$, with

$$F(x_k, P_n) = u_n(x_k) + \sum_{b \neq 0} P_n(b)a(b)x_k^{\frac{1}{2}k+n}K_{ir}(2\pi|b|x_k),$$

where $b = \{b_0, \cdots, b_{k-1}\}$ runs over the points of a k-dimensional lattice, and $P_n(b) = P_n(b_0, b_1, \cdots, b_{k-1})$ is any spherical harmonic of degree n. The Mellin transformation leads to Dirichlet series of the type $\sum_{b \neq 0} P_n(b)a(b)/|b|^{2s+n}$, with functional equations whose Γ-factor is $\Gamma(s + \frac{1}{2}(n + ir))\Gamma(s + \frac{1}{2}(n - ir))$. The paper does not contain any details.

H. Rademacher (Philadelphia, Pa.).

✱Severi, Francesco. **Funzioni quasi abeliane.** Pontificiae Academiae Scientiarum Scripta Varia, v. 4, Vatican City, 1947. 327 pp.

This memoir deals with all types of one-valued analytic functions of $\tau (\tau \geq 1)$ complex variables which possess an addition theorem, in the Weierstrass sense; such functions are therefore meromorphic at finite distance, and possess $\mu \leq 2\tau$ independent period-vectors. When $\mu = 2\tau$, such functions are the Abelian functions of τ variables. Functions with $\mu > 2\tau$ periods may be reduced to those with $\mu = 2\tau$ periods.

The first and second parts of the paper deal with that class of quasi-Abelian functions of τ variables which the author calls special, namely functions obtained from the inversion problem on a curve C of virtual genus τ and genus p, on which are assigned δ_1 pairs of distinct points, and δ_2 pairs of coincident points, so that $\tau = p + \delta_1 + \delta_2$. The author calls "neutral domain (campo neutro) on C" the set of all linear series on C, such that the above chosen $\delta = \delta_1 + \delta_2$ pairs are neutral pairs. Subsequently the author develops the "geometry" on C from both the algebraic-geometric and transcendental points of view. There are analogies and differences between this theory and the classical case: for instance, the Jacobi variety V_τ turns out to be the product of a linear space S_δ and of the Jacobi variety related to C (C being now considered as a curve of genus p) and possesses, as in the classical case, an Abelian group ∞^τ of birational transformations in itself, but this group is no longer transitive, but only transitive in general. The central point of this part is the inversion problem: τ simple integrals $u_1, \cdots, u_\tau$ are first constructed on V_τ, starting from τ linearly independent neutral integrals of the first kind on C; this enables one to think of the general point of V_τ, or also any rational function on V_τ, as a quasi-Abelian function of $u_1, \cdots, u_\tau$. By a suitable choice of the neutral integrals of the first kind on C, the matrix of the periods of the quasi-Abelian functions can be reduced to the form

$$(1) \qquad \begin{Vmatrix} A & \Omega & 0 \\ 0 & \Omega_1 & B \\ 0 & \Omega_2 & 0 \end{Vmatrix},$$

where the first and second columns are matrices with p columns, and the third column is a matrix with δ_1 columns;

while the first, second and third rows are, respectively, p-, δ_1-, and δ_2-rowed matrices; moreover, A, B are scalar matrices with $2\pi i$ as diagonal elements, and $\|A\Omega\|$ is an Abelian matrix connected with C. This starting point provides a useful point of view for the general study.

In the third part of the paper, the author points out that three classes of fields can be discussed, in order to begin the study of quasi-Abelian functions: (1) fields consisting of all meromorphic (at finite distance) functions of τ variables, whose period-matrix can be reduced to the form (1) (where now $\|A\Omega\|$ stands for any normal Riemann matrix); (2) fields consisting of all rational functions on an algebraic V_τ, possessing a group, transitive in general, of ∞^τ birational transformations in itself; (3) fields consisting of all rational functions on a V_τ, which is the product of a linear space S_δ and a Picard V_p, where $p + \delta = \tau$. The author succeeds in proving the equivalence of definitions (1), (2), (3), under a hypothesis ("hypothesis L") which, up to now, seems to be nontrivial; in the course of the proof, which is long and difficult, it is necessary to carry out a thorough examination of some general properties of the basis on an algebraic variety, and the study of the singular and indetermination manifolds of the simple integrals on V_τ. A V_τ with an Abelian group Γ, ∞^τ, transitive in general, possesses infinitely many such groups; this makes it necessary to attach the function-field not only to V_τ, but to the choice of Γ as well. It then becomes possible to count the number of moduls on which the fields of quasi-Abelian functions depend.

The next step leads to the highest part of the theory, i.e., to the structure theorem and the existence theorems. The first enables one to express every function of the field by means of rational, exponential, intermediary and related functions; the proof depends on generalizations of classical theorems on elliptic integrals to the simple integrals of the second and third kinds on a variety; such generalizations hold true for any variety, independently of the theory of quasi-Abelian functions. The most extensive existence theorem is the third and last, which states that a necessary and sufficient condition in order that a field of quasi-Abelian functions with a period-matrix of the form (1) exist, is that the matrix $\|A\Omega\|$ be a normal Riemann matrix, while no condition at all is imposed upon Ω_1 and Ω_2.

The fourth part of the memoir deals with quasi-Abelian functions of two variables. For a field of such functions, the author studies fully the surfaces which correspond to the Kummer surface in the classical case, and which may be considered as limiting cases of the latter. As an instance of such a limit surface, there is found a surface of the fourth order, which was also considered by Plücker, in connection with some research on the geometry of the straight line.

F. Conforto (Rome).

Theory of Series

Hsu, L. C. **Note on an asymptotic expansion of the nth difference of zero.** Ann. Math. Statistics 19, 273–277 (1948).

An asymptotic formula for Stirling numbers of the second kind (divided differences of zero), moving on diagonals in the ordinary triangular array, is given in the form

$$S_{n, n+k} = \frac{n^{2k}}{2^k k!}\left\{1 + \frac{f_1(k)}{n} + \cdots + \frac{f_l(k)}{n^l} + O(n^{-l-1})\right\}$$

and the functions f_1, f_2 and f_3 are given explicitly. This, with Stirling's formula, since $n!S_{n, n+k} = \Delta^n 0^{n+k}$, gives a formula of similar extent for the differences of zero themselves.

$J.\ Riordan$ (New York, N. Y.).

Birindelli, Carlo. **Una rapida trattazione della serie binomiale nel campo complesso.** Matematiche, Catania 2, 136–155 (1947).

Rollero, Aldo. **Su un criterio di convergenza per le serie a termini reali positivi.** Euclides, Madrid 8, 5–8 (1948).

Four simple series are tested for convergence by use of the elementary fact that $\sum u_n$ diverges if $nu_n \to L \neq 0$ and $\sum v_n$ converges if $s > 1$ and $n^s v_n \to M$. $R.\ P.\ Agnew.$

Rollero, Aldo. **Sul calcolo grafico di un limite.** Atti Accad. Ligure 3 (1943), 277–282 (1946).

Discussion of $\lim x_n$, where $x_1 = 1$ and $x_n = m^{x_{n-1}}$, by the usual diagram for iteration ($y = x$ and $y = m^x$). If $m < 1$, then $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are monotone. If $\lim x_n$ exists, their limits coincide (as for $m = \frac{1}{3}$). Otherwise there exists a λ so that $\lambda^{m^\lambda} = \lambda$. For $m > 1$, $\lim x_n = \infty$. $J.\ W.\ Tukey.$

Wintner, Aurel. **A sequence of Weierstrassian summations.** J. London Math. Soc. 22 (1947), 311–314 (1948).

The author states that the Weierstrass device, for inserting the familiar special factors into infinite products to produce convergent infinite products, suggests the following definition of specific methods $(W, 0)$, $(W, 1)$, $\cdots$ of summability. Let k be a nonnegative integer, and let

$$W_k(z) = (1 + z + z^2/2! + \cdots + \frac{z^k}{k!})e^{-z}.$$

A series $a_1 + a_2 + \cdots$ is summable (W, k) to L if the series in $f_k(s) = \sum_{n=1}^{\infty} W_k(ns)a_n$ converges when $s > 0$ and defines a function $f_k(s)$ such that $f_k(s) \to L$ as $s \to 0$. Then $(W, 0)$ is the Euler-Abel power series method of summability. If k is a positive integer and $\sum a_n$ is summable (W, k), then it is also summable $(W, k-1)$ to the same value. For each $k = 0, 1, 2, \cdots$, (W, k) is a regular method of summability which includes all Cesàro methods C_r of positive orders r. $R.\ P.\ Agnew$ (Ithaca, N. Y.).

Schurr, Zvi. **On absolute regularity of linear transformations.** Riveon Lematematika 2, 12–17 (1947). (Hebrew)

Hahn [Monatsh. Math. Phys. 32, 3–88 (1922)] treated sequence-to-sequence transformations of the form

$$(1) \qquad \sigma_n = \sum_{k=0}^{\infty} a_{nk} s_k.$$

He dealt, in particular [§ 6, pp. 21–27], with sequences having bounded variation; a sequence s_n has bounded variation if $\sum |s_n - s_{n-1}| < \infty$. Hahn showed that (1) carries each sequence s_n having bounded variation into a convergent sequence σ_n if and only if there exist constants M, α_0, α_1, α_2, $\cdots$, and ρ such that $(2)\ |\sum_{k=0}^{\infty} a_{nk}| \leq M$ $(n, k = 0, 1, 2, \cdots)$, $(3)\ \lim_{n \to \infty} a_{nk} = \alpha_k$ $(k = 0, 1, 2, \cdots)$, and $(4)\ \lim_{n \to \infty} \sum_{k=0}^{\infty} a_{nk} = \rho$; and, moreover, if these conditions (2), (3), and (4) hold and if the sequence s_n converges to s and has bounded variation, then $(5)\ \lim_{n \to \infty} \sigma_n = \rho s + \sum_{k=0}^{\infty} \alpha_k(s_k - s)$.

F. M. Mears [Ann. of Math. (2) 38, 594–601 (1937)] characterized the smaller class S of transformations (1) which carry each sequence s_n having bounded variation into a sequence σ_n having bounded variation; these transformations are called absolutely conservative.

Citing the work of Mears but not that of Hahn, the author shows that if (1) belongs to the smaller class S, and if the sequence s_n converges to s and has bounded variation, then (5) holds. Thus the author joins the considerable number of authors who have rediscovered parts of the results of Hahn's § 6. One who wishes a straightforward proof of the author's result may observe that if (1) is absolutely conservative, then it must, a fortiori, satisfy Hahn's conditions; use of the Abel partial summation formula then gives (5). $R.\ P.\ Agnew$ (Ithaca, N. Y.).

Racine, C. **On Frullani integrals.** J. Indian Math. Soc. (N.S.) 11, 95–97 (1947).

K. S. K. Iyengar [same J. (N.S.) 4, 145–150 (1940); Proc. Cambridge Philos. Soc. 37, 9–13 (1941); these Rev. 2, 219, 218] showed that if $\varphi(t)$ is integrable over each finite interval $a \leq t \leq b$ for which $a > 0$, then (1) $\lim_{s \to 0} \int_{\epsilon s}^{s} t^{-1} \varphi(t)dt$ exists for each $\rho > 0$ if and only if the two limits (2) $\lim_{u \to 0} \int_{u}^{1} \varphi(t)dt$, $\lim_{u \to 0+} u^{-1} \int_{0}^{u} \varphi(t)dt$ both exist. Racine simplifies a part of Iyengar's proof. When one sets $\rho = e^{-\lambda}$, $\epsilon = e^{-A}$, $u = e^{-t}$, and $\varphi(e^{-t}) = f(t)$, the limit in (1) takes the form (3) $\lim_{A \to \infty} \int_{A}^{A+\lambda} f(u)du$. Using the form (3), the reviewer [Duke Math. J. 9, 10–19 (1942); these Rev. 3, 233] generalized and simplified Iyengar's work. As the reviewer has learned subsequently, some of the results of Iyengar, as well as some of those of the reviewer, are easy deductions from a theorem of Plancherel and Pólya [Comment. Math. Helv. 3, 114–121 (1931)]: if

$$\lim_{R \to \infty} (2R)^{-1} \int_{x-R}^{x+R} f(u)du = \varphi(x)$$

exists for each real x, then $\varphi(x)$ is a linear function of x.

$R.\ P.\ Agnew$ (Ithaca, N. Y.).

Fourier Series and Generalizations, Integral Transforms

Stečkin, S. B. **A generalization of some inequalities of S. N. Bernštein.** Doklady Akad. Nauk SSSR (N.S.) 60, 1511–1514 (1948). (Russian)

Nikol'skiĭ, S. **A generalization of an inequality of S. N. Bernštein.** Doklady Akad. Nauk SSSR (N.S.) 60, 1507–1510 (1948). (Russian)

Bernštein, S. N. **A generalization of an inequality of S. B. Stečkin to entire functions of finite degree.** Doklady Akad. Nauk SSSR (N.S.) 60, 1487–1490 (1948). (Russian)

Let $t_n(x)$ be a trigonometric polynomial of degree n and $\Delta_\delta^r t_n(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} t_n(x+j\delta)$. The first paper proves the inequality

$$(*) \qquad |t_n^{(r)}(x)| \leq (\tfrac{1}{2}n \csc \tfrac{1}{2}n\delta)^r \sup_s |\Delta_\delta^r t_n(x)|, \qquad 0 < \delta < 2\pi/n.$$

The case $r = 1$ is deduced from the lemma that for $|\eta| \leq \pi/n$ and $t_n(x_0) = L = \sup |t_n(x)|$, we have $t_n(x_0 + \eta) \geq L \cos n\eta$. The general case follows by induction on r. Similar extensions of classical theorems on ordinary polynomials are pointed out.

The second paper extends (*) to entire functions of exponential type n (no longer necessarily an integer) in the case $\delta = \pi/n$ and then to functions of more than one variable.

The third paper gives a different proof of (*) for $r = 1$ and entire functions of exponential type, and adds the inequality

$$(**) \qquad |t_n(x+\delta) - t_n(x)| \leq 2 \sin \tfrac{1}{2}n\delta \cdot \sup_s |t_n(x)|, \qquad 0 < \delta < \pi/n.$$

The reviewer remarks that (*) and (**) for finite Fourier-Stieltjes integrals follow at once from a theorem of P. Civin [Duke Math. J. 8, 656–665 (1941); these Rev. 3, 108]. For the more general class of entire functions of exponential type they may then be deduced by a simple limiting process.

R. P. Boas, Jr. (Providence, R. I.).

Kuttner, B. A further note on the Gibbs phenomenon. J. London Math. Soc. 22 (1947), 295–298 (1948).

The author considers methods of summability in which the sum $\sum_{n=0}^{\infty}a_n$ is defined by $\lim_{u\to\infty}\sum_{n=0}^{\infty}q(n/u)a_n$ with $q(x)$ continuous, absolutely integrable, of bounded variation in $(0, \infty)$ and with $q(0)=1$. If the kernels of such methods in the summability of Fourier series are bounded below, they are shown to be nonnegative. The earlier result of the author [same J. 20, 136–139 (1945); these Rev. 7, 518] then implies that for such methods of summability a necessary and sufficient condition that the Gibbs phenomenon should not occur is that the kernel be positive. Riesz summability (R, n^λ, k), Abel summability (A, n^λ) and Riemann summability come within the scope of the work for suitable λ.

P. Civin (Eugene, Ore.).

Wang, Fu Traing. Summability of Fourier series by Riesz's logarithmic means. III. Duke Math. J. 15, 5–10 (1948).

Let $f(t)$ be an integrable function of period 2π with Fourier series (*) $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nt+b_n\sin nt)$. Let

$$\varphi(t)=\frac{1}{2}(f(x+t)+f(x-t)-2S)$$

and

$$\Gamma(\alpha)\varphi_\alpha(t)=\int_0^t (t-u)^{\alpha-1}\varphi(u)du.$$

In papers I and II [Tôhoku Math. J. 40, 142–159, 274–292 (1935)] the author demonstrated that for α a positive integer and (**) $\varphi_\alpha(t)=o(t^\alpha)$ as $t\to0$ then (*) is summable by Riesz's logarithmic means at $t=x$ to S. The present paper demonstrates the same result for $\alpha>1$ and also shows that the condition (**) is insufficient if $0<\alpha<1$.

P. Civin (Eugene, Ore.).

Chow, H. H. On the summability for negative indices of the Fourier series of a monotonic function with an infinite limit. J. London Math. Soc. 22 (1947), 262–268 (1948).

Let $f(\theta)$ be a periodic function of class L and $\sigma_n{}^r(\theta)$ the Cesàro mean of order r of the Fourier series of $f(\theta)$. Let $\phi(t)=\frac{1}{2}\{f(\theta+t)+f(\theta-t)\}$. Then there exists a positive constant $\rho<1$ such that (i) if $r>-\rho$, $\phi(t)$ is monotonic and $\lim_{t\to0}\phi(t)=+\infty$ then $\lim_{n\to\infty}\sigma_n{}^r=+\infty$, (ii) if $-1<r<-\rho$ there exist monotonic $\phi(t)$ with $\lim_{t\to0}\phi(t)=+\infty$ for which $\liminf_{n\to\infty}\sigma_n{}^r$ is not $+\infty$.   K. Chandrasekharan.

Cheng, Min-Teh. Summability factors of Fourier series. Duke Math. J. 15, 17–27 (1948).

Let $\sum(a_n\cos nx+b_n\sin nx)$ be the Fourier series of a Lebesgue integrable function. Generalizing results of J. M. Whittaker (with $\lambda_n=n^{-\epsilon}$, $\epsilon>0$), B. N. Prasad (with $\lambda_n=(\log n)^{-1-\epsilon}$) and S. Izumi and T. Kawata, for summability $|A|$, H. C. Chow proved that, if λ_n is convex and such that $\sum n^{-1}\lambda_n$ converges, then (*) $\sum\lambda_n(a_n\cos nx+b_n\sin nx)$ is summable $|C,1|$ almost everywhere [Whittaker, Proc. Edinburgh Math. Soc. (2) 2, 1–5 (1930); Prasad, Proc. London Math. Soc. (2) 35, 407–424 (1933); Izumi and Kawata, Proc. Imp. Acad. Tokyo 14, 32–35 (1938); Chow, J. London Math. Soc. 16, 215–220 (1941); these Rev. 4, 37].

This however did not establish the behaviour of (*) throughout the Lebesgue set.

Here the author shows that, in the Lebesgue set, (1) with $\lambda_n=(\log n)^{-1-\delta}$ the series (*) is summable $|C, 1+\delta|$, $\delta>0$; (2) with $\lambda_n=(\log n)^{-1-\epsilon}$ it is summable $|C, 1|$; (3) with $\lambda_n=n^{-\alpha}$, $0<\alpha\le1$, it is summable $|C, 1-\alpha+\delta|$; (4) with $\lambda_n=n^{-\alpha}(\log n)^{-1-\epsilon}$ it is summable $|C, 1-\alpha|$.

Result (1) is proved by a direct argument. There is a similar result for conjugate series. Result (2) is deduced from a result of G. H. Hardy and J. E. Littlewood on strong summability [Fund. Math. 25, 162–189 (1935)]. The author states that a similar result for conjugate series may be proved "in a similar manner." But the improved lemma 3 (part 2), on which it is said to depend, is false as it stands; it should be pointed out that only part 1 of lemma 3 was given by Hardy and Littlewood. Result (3) is proved by a direct argument. It may also be deduced from (1) by a theorem of E. Kogbetliantz [Bull. Sci. Math. (2) 49, 234–256 (1925)]. For (*) is summable $|C, 1+\eta|$, $\eta>0$, with $\lambda_n=n^{-\epsilon}$, and hence summable $|C, 1-\alpha+\eta+\epsilon|$ with $\lambda_n=n^{-\alpha}$, $\alpha>\epsilon$.

The proof of (4) is also by a direct argument. The reviewer, however, cannot follow the statement that lemma 5 can be proved by "the same argument as is the proof of lemma 2," and is only able to complete the proof by using his own analogue of Riesz's mean value theorem [Bosanquet, J. London Math. Soc. 16, 146–148 (1941); these Rev. 3, 144].

L. S. Bosanquet (London).

Cheng, Min-Teh. Summability factors of Fourier series at a given point. Duke Math. J. 15, 29–36 (1948).

It is known that, if $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, then the Fourier series of $f(x)$ is summable $|C, \alpha|$, where $\varphi_0(t)=\varphi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$, $|C, \alpha+\delta|$, $\delta>0$

$$\varphi_\alpha(t)=\alpha t^{-\alpha}\int_0^t (t-u)^{\alpha-1}\varphi(u)du,   \alpha>0$$

[Bosanquet, J. London Math. Soc. 11, 11–15 (1936); Proc. London Math. Soc. (2) 41, 517–528 (1936)]. Also if $\varphi(t)$ is of bounded variation in $(0, \pi)$, then

$$A_n(x)=a_n\cos nx+b_n\sin nx=O(n^{-1}),$$

and hence (*) $\sum A_n(x)(\log n)^{-1-\epsilon}$ is absolutely convergent. Here the author proves directly that, if $\varphi_\alpha(t)$ is of bounded variation in $(0, \pi)$, $0<\alpha\le1$, then (*) is summable $|C, \alpha|$.

L. S. Bosanquet (London).

Šerbina, A. D. On a generalization of the method of Fejér for the summation of a double Fourier series. Doklady Akad. Nauk SSSR (N.S.) 60, 1321–1324 (1948). (Russian)

It is well known [cf. L. Tonelli, Serie Trigonometriche, Zanichelli, Bologna, 1928, chap. IX, pp. 490–494] that the Fejér means

$$\sigma_{m,n}=\frac{1}{(m+1)(n+1)}\sum_{\mu=0}^{m}\sum_{\nu=0}^{n}S_{\mu\nu}$$

of the double Fourier series of a continuous function $f(x, y)$, periodic with period 2π with respect to both variables, converge uniformly to the function. The aim of the paper is to extend this result to the generalized Fejér means

(1) $$\sigma_{m,n,p,q}=\frac{1}{(p+1)(q+1)}\sum_{\mu=m-p}^{m}\sum_{\nu=n-q}^{n}S_{\mu\nu},$$

where $p=p(m)$ $(0\le p\le m)$ and $q=q(n)$ $(0\le q\le n)$ are func-

tions of m and n, respectively. It is stated that the conditions

$$(2) \qquad \liminf_{m \to \infty} p(m)/m = \alpha > 0, \quad \liminf_{n \to \infty} q(n)/n = \beta > 0$$

are necessary and sufficient for $\sigma_{m,n,p,q}$ to tend uniformly to $f(x, y)$ when $m, n \to \infty$. Let $M^{m,n}_{p,q}$ denote the Lebesgue constants of the summation method (1). It follows from a result of S. Nikolsky [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 4, 509–520 (1940); these Rev. 2, 279] that

$$(3) \qquad M^{m,n}_{p,q} = \frac{16}{\pi^4} \log \frac{m}{p+1} \cdot \log \frac{n}{q+1} + O\left(\log \frac{m}{p+1}\right)$$
$$+ O\left(\log \frac{n}{q+1}\right) + O(1).$$

Thus if conditions (2) are not satisfied it is seen from (3) that the Lebesgue constants are not bounded, which proves the necessity of the conditions. Now let $E_{m-p,\,n-q}(f)$ denote the maximal deviation from $f(x, y)$ of the best approximating trigonometric polynomial of order $m-p$, $n-q$. As we have

$$(4) \qquad |\sigma_{m,n,p,q} - f(x, y)| \leqq (M^{m,n}_{p,q} + 1) E_{m-p,\,n-q}(f);$$

furthermore, as the Lebesgue constants $M^{m,n}_{p,q}$ are bounded if condition (2) is fulfilled, it follows from (4) that $\sigma_{m,n,p,q}$ converges uniformly to $f(x, y)$ provided that $m-p \to \infty$ and $n-q \to \infty$. Now it seems, though it is nowhere stated explicitly, that the author assumes that $m-p(m)$ and $n-q(n)$ are nondecreasing functions of their arguments [for instance, it is stated on p. 1323 that if in (2) $\alpha < 1$ and $\beta < 1$ we have $m-p(m) \to \infty$ and $n-q(n) \to \infty$, which of course is true only under the additional condition mentioned]. In case either $m-p(m)$ or $n-q(n)$ or both are bounded, the proof follows by comparing $\sigma_{m,n,p,q}$ with the ordinary Fejér means $\sigma_{m,n}$. The same method furnishes the proof also in case the existence of $\lim_{m \to \infty} p(m)/m$ is supposed.

[Reviewer's remark. The theorem in question can be proved without the limitations that $m-p(m)$ and $n-q(n)$ are nondecreasing, and without making distinctions between different cases, by the same well-known argument [see, for instance, Tonelli, loc. cit.] by which the uniform convergence of the ordinary Fejér sums is usually proved. As a matter of fact (2) implies that the Lebesgue constants are bounded, furthermore that $p \to \infty$ and $q \to \infty$, and this is all that is needed.]
A. Rényi (Budapest).

Bosanquet, L. S. On convergence and summability factors in a Dirichlet series. J. London Math. Soc. 22 (1947), 190–195 (1948).

Il est bien connu que de la convergence de la série
$$(1) \qquad \sum a_n l_n^{-s} \text{ pour } s = s_0 \text{ il suit la convergence absolue de}$$
la même série pour $\Re(s) > \Re(s_0) + D$, où
$$(2) \qquad D = \limsup_{n \to \infty} \log n / \log l_n.$$

L'auteur généralise ce résultat pour la sommabilité absolue $|R, l, \kappa|$, en démontrant le théorème suivant. Soit κ un nombre entier positif et soit $\sum a_n$ bornée (R, l, κ). Alors la série (1) est sommable $|R, l, \kappa|$ pour $\Re(s) > D$, où D est déterminé par (2).
N. Obrechkoff (Sofia).

✱**McLachlan, N. W.** Modern Operational Calculus with Applications in Technical Mathematics. Macmillan and Co., Ltd., London, 1948. xiv+218 pp. $5.00.

This is a book on the theory and applications of the Laplace transformation directed especially to graduate engi-

neers and technologists. It will be of interest to pure and applied mathematicians as well. In the mathematical analysis, constant attention is paid to questions of convergence of infinite series and integrals, of the change of order of integration and other matters of rigor. The six chapters making up the body of the book carry the following headings: The Laplace transform, Theorems or rules of the operational calculus, Solution of ordinary linear differential equations with constant coefficients, Evaluation of integrals and establishment of mathematical relationships, Derivation of Laplace transforms of various functions, Laplace transforms for a finite interval: Impulses. The appendices include brief discussions of Heaviside's unit function, the convergence of infinite series and integrals and the inversion integral. Throughout the book, the properties of the transformation are copiously illustrated. Many of the illustrative examples deal with Bessel functions. The physical applications treated include problems in electrical circuits, mechanical systems, transmission lines, compressional shock waves in fluids and propagation of sound in viscous media. The 250 exercises given toward the end of the book include problems on convergence, the derivation of special properties of transformations, the derivation of transforms and other properties of special functions, of Bessel functions in particular, as well as the solution of problems in ordinary and partial differential equations and integral equations. The book ends with a list of about 80 transforms and a short bibliography. Although the inversion integral and residue theory are used to some extent, the author leaves the extensive use of that method to his earlier volume on complex variables and operational calculus. Most of the inversions made in the present book are performed with the aid of short tables of transforms. *R. V. Churchill*.

✱**Humbert, Pierre, et Colombo, Serge.** Le calcul symbolique et ses applications à la physique mathématique. Mémor. Sci. Math., no. 105. Gauthier-Villars, Paris, 1947. 52 pp.

This is an introduction to operational calculus and its applications, written for technologists and physicists, without any attempt at mathematical rigour, and with emphasis on practical methods. The introduction contains historical remarks [but, like most other similar books, fails to mention H. Bateman] and illustrates the "spirit of Heaviside's operational calculus" by a few examples. After this, the Heaviside calculus in its original form is left aside, and all work is based on the "Carson integral."

Part I contains the rules of the symbolic calculus; $\varphi(p) = p \int_0^\infty \epsilon^{-pt} f(t) dt$ is called the "image" of the "original" $f(t)$. N. W. McLachlan's notation $\varphi(p) \subset f(t)$ is used. Next the well-known rules of the correspondence $\varphi(p) \subset f(t)$ are derived. In addition to the elementary rules there are a few useful "symbolic sequences," such as: if $\varphi(p) \subset f(t)$ and $p^{\frac{1}{2}} f(1/p) \subset h(t)$ then $\varphi(p^2) \subset \frac{1}{2}\pi^{\frac{1}{2}} t h(\frac{1}{4}t^2)$. The evaluation of images of a few elementary functions follows, and the Bromwich-Wagner integral (complex inversion formula) closes this part.

Part II illustrates the application of the symbolic calculus to the investigation of special functions. Bessel functions of the first kind, Kelvin's ber and bei functions, the exponential integral and related functions, Fresnel's integrals, Gilbert's $P_\lambda(p) = \int_0^\infty \epsilon^{-px} x^{\lambda-1}(1+x^2)^{-\frac{1}{2}} dx$, the function $\nu(t) = \int_0^\infty t^s / \Gamma(s+1) ds$, and periodic functions are treated.

Part III contains applications to ordinary linear differential equations with constant coefficients, to electrical

Gurland, John. **Inversion formulae for the distribution of ratios.** Ann. Math. Statistics 19, 228–237 (1948).

Let $F(x)$ be a distribution function and $\phi(t)$ its characteristic function. Then $F(x) + F(x-0) = 1 - (\pi i)^{-1} \int_{-\infty}^{\infty} t^{-1} e^{-ixt} \phi(t) dt$, the integral being a principal value at 0 and ∞. This variant of the inversion formula is used to obtain the distribution of the ratio $\sum_1^n a_j X_j / \sum_1^n b_j X_j$, $X_1, \cdots, X_n$ being random variables whose joint distribution is known, and the a's and b's real constants. The inversion formula is generalised to n variables and yields the joint distribution of several ratios, and the joint distribution of several ratios of quadratic forms in random variables having a multivariate normal distribution.   *G. E. H. Reuter* (Manchester).

Agarwal, R. P. **On the resultant of two functions.** Proc. Indian Acad. Sci., Sect. A. **27**, 141–146 (1948).

The author defines two functions $f_1(x)$ and $f_2(x)$ by means of infinite integrals and proves that their resultant, $g(x) = \int_0^\infty f_1(y) f_2(xy) dy$, is a kernel which transforms an R_λ into an R_μ. In particular, if f_1 and f_2 are R_ρ and R_ν, respectively, their resultant transforms an R_ρ into an R_ν. Some examples are included.   *M. C. Gray* (New York, N. Y.).

Poli, L. **Sur les fonctions réciproques.** Ann. Univ. Lyon. Sect. A. (3) **10**, 23–38 (1947).

In this paper the investigation of pairs of Hankel transforms by means of the operational calculus is based on two relations of which one is [correctly] ascribed to F. Tricomi, and the other is thought to be new. [It is, however, due to H. C. Gupta, J. Indian Math. Soc. (N.S.) 7, 117–128 (1943), theorem 2; these Rev. 6, 50]. The conditions of validity are not stated fully ["Les conditions sont difficiles à préciser à cause du caractère oscillant de $J_n(z)$ lorsque $z \to \infty$"]. Examples of pairs of Hankel transforms, of the usual type, are given.   *A. Erdélyi* (Edinburgh).

Zamansky, Marc. **Sur l'approximation des fonctions continues.** C. R. Acad. Sci. Paris 226, 1066–1068 (1948).

Soit $f(x)$ une fonction continue, de période 2π, et soit $P_n(x)$ un polynome trigonométrique d'ordre n tel que

$$|P_n(x) - f(x)| \leq \varphi(n)/n^{k-1}, \qquad n=1, 2, \cdots,$$

où $\varphi(x) > 0$, $\varphi(x) \to 0$ en décroissant lorsque $x \to \infty$, k étant un entier positif. En appliquant le théorème de Bernstein, on peut montrer que, dans ces conditions,

$$|P_n^{(k)}(x)| \leq A + n\varphi(n) + B \int_{n_0}^n \varphi(x) dx,$$

où A est une constante ne dépendant que de n_0 et B est une constante universelle. Ce résultat contient ceux démontrés précédemment par l'auteur [mêmes C. R. **224**, 704–706 (1947); ces Rev. 8, 457]. Dans le cas où $|P_n(x) - f(x)| \leq K\omega(1/n)$, $\omega(h)$ étant le module de continuité de $f(x)$, on a $|P_n'(x)| \leq Cn\omega(1/n)$, où C est une constante indépendante de x et n. On trouve ensuite une série de théorèmes sur les relations entre la meilleure approximation E_n de $f(x)$ par un polynome trigonométrique d'ordre n, et du module maximum μ_n de la dérivée du polynome de meilleure approximation.

Enfin, on énonce le théorème suivant. Si la fonction $f(x)$ admet une dérivée finie en tout point de l'intervalle (a, b),

et si $\{P_n(x)\}$ est une suite de polynomes tendant vers $f(x)$ uniformément sur (a, b), alors $f'(x)$ est un point d'accumulation de $\{P_n'(x)\}$ pour tout x sur un résiduel de (a, b).
                    B. de Sz. Nagy (Szeged).

Korevaar, J. **A theorem on uniform approximation.** Simon Stevin **25**, 201–207 (1947).

Reprinted from Nederl. Akad. Wetensch., Proc. 49, 752–757 (1946); these Rev. 8, 140.

Hartman, Philip, and Wintner, Aurel. **Töplerian (L^2)-bases.** Trans. Amer. Math. Soc. 63, 207–225 (1948).

Suppose $f(x)$ is odd, periodic of period 2π in $L_2 = L_2(-\pi, \pi)$. This paper considers the two problems: (I) when is $\{f(nx)\}$ complete in L_2 and (II) when does $\{f(nx)\}$ yield an expansion for every element of L_2? If $f(x) \sim \sum a_n \sin nx$ then call $\phi(s) = \sum a_n n^{-s}$ the associated Dirichlet series. Write $M(\psi(s))$ for $\lim_{t \to \infty} (2t)^{-1} \int_{-t}^{t} |\psi(i\tau)|^2 d\tau$. We give first a rough classification of methods available for these problems: (A) use of a matrix due to Toeplitz; (B) use of the infinite linear equation system stemming from the equivalent requirements of completeness and closure; (C) for cases such as $a_n = n^{-\lambda}$, the detailed bounding of Möbius sums; (D) use of the principle that $\lim_{N \to \infty} M\varphi(s)\psi_N(s) = 1$ for suitable $\psi_N(s)$ in some cases when $M(\varphi^{-1})$ is meaningless. These methods all involve the associated Dirichlet series essentially. Let (E) stand for methods not directly of this type. The authors here and separately in other recent publications [Wintner, Quart. J. Math., Oxford Ser. 18, 209–214 (1947); Hartman, Duke Math. J. 14, 755–767 (1947); these Rev. 9, 346, 426] use (A), (B), (C), with special stress on (A) and (B). The reviewer has used (A), (D) and (E) with special stress on (D). [Apparently the writers became aware of the reviewer's paper [same Trans. 60, 478–518 (1946); these Rev. 8, 512] in the course of reading proof, but their reference to the reviewer's methods is misleading, and they overlooked the fact that many of their key theorems already occur in this paper. For instance, not only are emphasis and examples on the distinction between (I) and (II) already in this paper, but also except for trivial differences, it contains the following of the authors' theorems: 8, 9, 14 and 17. Their theorem 7 is to be found in a paper by Bourgin and Mendel [same Trans. **57**, 332–363 (1945); these Rev. 6, 266].] The paper's 23 theorems include some determinations of the pertinent properties of the associated Dirichlet series [cf. the review of the paper by Hartman]. [In theorem 14 the condition $\varphi_1 = 0$ should obviously be $\varphi_1 \neq 0$.]   *D. G. Bourgin.*

Polynomials, Polynomial Approximations

Erdös, Paul, and Niven, Ivan. **On the roots of a polynomial and its derivative.** Bull. Amer. Math. Soc. 54, 184–190 (1948).

It was shown by de Bruijn [Nederl. Akad. Wetensch., Proc. 49, 1037–1044 = Indagationes Math. 8, 635–642 (1946); these Rev. 8, 377] that, if $r_1, \cdots, r_n$ are the zeros of a real polynomial $f(z)$ and $R_1, \cdots, R_{n-1}$ those of its derivative, then

$$(1) \qquad n^{-1} \sum_{j=1}^{n} |\Im(r_j)| \geq (n-1)^{-1} \sum_{j=1}^{n-1} |\Im(R_j)|.$$

Subsequently he and Springer showed [Nederl. Akad. Wetensch., Proc. 50, 264–270 = Indagationes Math. 9, 458–

464 (1947); these Rev. 9, 30] that (1) is valid also for complex polynomials. The present article establishes the same fact, the proof having been developed independently in the interim between the publication of the above-mentioned articles. The present proof is obtained by induction upon the number of zeros of $f(z)$ is the lower half-plane.

M. Marden (Milwaukee, Wis.).

Ballieu, Robert. **Sur des limitations des racines d'une équation algébrique.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 743–750 (1947).

The principal result of this paper is that, if $A_0=1$, $A_n=0$ and if A_j, $j=1, 2, \cdots, n-1$, are arbitrary positive numbers, then the real equation $x^n = \sum_1^n a_j x^{n-j}$ has all its positive roots (if any) on the interval

$$\min \left[(A_j+a_j)/A_{j-1}\right] \leqq x \leqq \max \left[(A_j+a_j)/A_{j-1}\right],$$
$$j=1, 2, \cdots, n.$$

This result follows by requiring t to satisfy expressions such as

$$(a_j+A_j)t \leqq 1; \quad (a_j+A_j)t \leqq A_{j-1};$$
$$j=2, 3, \cdots, n-1; a_n t \leqq A_{n-1},$$

with the inequality sign holding in at least one of these expressions. [Reviewer's note: An immediate extension of this result is that all the roots of this equation, real or complex, lie in the circle $|x| \leqq \max B_j$, $j=1, 2, \cdots, n-1$, where $B_j=(A_j+|a_j|)/A_{j-1}$ and where the a_j are real or complex. When $A_j=1$, $j=1, 2, \cdots, n-1$, this result reduces to the classical theorem of Cauchy.] *M. Marden.*

Williams, J. **The distribution of the roots of a complex polynomial equation.** Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. **2238** (9323), 9 pp. (1946).

The author derives a formula for the number p of zeros of a polynomial $f(z)=z^n+a_1 z^{n-1}+\cdots+a_n$, $a_n \neq 0$, the a_j real or complex, in a sector $A<\theta<A'\leqq A+2\pi$. Setting $f(re^{i\theta})=S_0+S_1 i$ with S_0 and S_1 real polynomials in r, he forms the Sturm sequence $S(\theta)$: $S_0, S_1, \cdots, S_q$. Traversing the ray $\theta=A$ from $r=0$ to $r=\infty$, he determines the number $L(A)$ of variations of sign in the leading coefficients of the $S(A)$, the number $C(A)$ of variations of sign in the constant terms in $S(A)$ and the sum $m(A)$ of the Cauchy indices of S_0/S_1 at the zeros of $f(z)$ on the ray $\theta=A$. Traversing the ray $\theta=A'$ from $r=\infty$ to $r=0$, he determines for the sequence $S(A')$ the corresponding numbers $L(A')$, $C(A')$ and $m(A')$. Denoting also by N the total number of zeros of $f(z)$ on the rays $\theta=A$ and $\theta=A'$ and by E the number of times which a radius vector in rotating from nA to nA' crosses the imaginary axis, he finally sets up the formula

$$2p=L(A)-C(A)+C(A')-L(A')+E-m(A)+m(A')-N.$$

His proof is based upon the principle of argument and the usual properties of Sturm sequences. [Reviewer's note: similar results are given by S. Sherman, Philos. Mag. (7) **37**, 537–551 (1946); these Rev. 8, 579.] *M. Marden.*

Cremer, Hubert. **Über den Zusammenhang zwischen den Routhschen und den Hurwitzschen Stabilitätskriterien.** Z. Angew. Math. Mech. 25/27, 160–161 (1947).

This note calls attention to a generally well-known relation between the sequence of determinants used in the Hurwitz criterion for stability and the sequence of terms used in the Routh criterion for stability. *M. Marden.*

Cremer, L. **Ein neues Verfahren zur Beurteilung der Stabilität linearer Regelungs-Systeme.** Z. Angew. Math. Mech. 25/27, 161–163 (1947).

The author restates the theorem of Biehler [J. Reine Angew. Math. **87**, 350–352 (1879)] that all the zeros of the real polynomial $a_0 s^n+a_1 s^{n-1}+\cdots+a_n$ lie in the left half-plane if the zeros of the polynomials $a_n - a_{n-2} s^2+\cdots$ and $s(a_{n-1}-a_{n-3} s^2+\cdots)$ are all real and separate one another. He suggests the theorem as a criterion simpler in practice than the criteria of Routh, Hurwitz or Nyquist.

M. Marden (Milwaukee, Wis.).

Schmidt, Hermann. **Bemerkung zu der Arbeit von L. Cremer: Ein neues Verfahren zur Beurteilung der Stabilität linearer Regelungs-Systeme.** Z. Angew. Math. Mech. 28, 124–125 (1948).

This note contains two remarks on the paper reviewed above. The first is that an alternative graphical determination of the zeros of the polynomial $f(x+iy)=P(x, y)+iQ(x, y)$ is by means of the intersection points of the two curves $P(x, y)=0$ and $Q(x, y)=0$. The second is that, if C denotes the curve into which the function $w=f(z)$ maps the y-axis, the Biehler criterion is essentially one of finding the number of windings of C about the point $w=0$. *M. Marden.*

Baier, O. **Die Hurwitzschen Bedingungen.** Z. Angew. Math. Mech. 28, 153–157 (1948).

This paper contains the details for the elementary derivation of the Hurwitz criterion outlined previously by the author [Ber. Math.-Tagung Tübingen 1946, pp. 40–41 (1947); these Rev. 9, 30]. *M. Marden* (Milwaukee, Wis.).

Nassif, M. **On the product of simple series of polynomials.** J. London Math. Soc. 22 (1947), 257–260 (1948).

The sequence of polynomials $\{p_n(z)\}$ $(n=0, 1, \cdots)$ is a "basic set" if every polynomial is uniquely representable as a finite linear combination of terms of the sequence; and is "effective" in $|z| \leqq R$ if every function $f(z)$ regular in $|z| \leqq R$ is represented there by the associated basic series $\sum_0^\infty \Pi_n f(0) p_n(z)$. Here Π_n is the operator

$$\Pi_n = \sum_{j=0}^\infty \frac{1}{j!} \pi_{jn} \frac{d^j}{dz^j},$$

and π_{jn} is given by $z^n = \sum \pi_{nj} p_j(z)$. The following result is established. If the polynomial (basic) sets $p_n(z)=\sum_0^n p_{nj} z^j$, $q_n(z)=\sum_0^n q_{nj} z^j$, with $p_n(z)$, $q_n(z)$ of degree exactly n, are effective in $|z| \leqq R$, and if $q_{nn}=1$, then the "product set" $\{u_n(z)\}$, defined by $u_n(z)=\sum_0^n p_{nj} q_j(z)$, is also effective in $|z| \leqq R$. The proof makes use of a condition for effectiveness due to Cannon [Proc. London Math. Soc. (2) **43**, 348–365 (1937)]. *I. M. Sheffer* (State College, Pa.).

Mandelbrojt, Szolem. **Sur l'approximation polynomiale des fonctions sur tout l'axe réel.** C. R. Acad. Sci. Paris 226, 1668–1670 (1948).

The author's theorem is a generalization of a well-known one by S. Bernstein [Leçons sur les Propriétés Extrémales, Gauthier-Villars, Paris, 1926, p. 62]. Let $\{\nu_n\}$ be an increasing sequence of positive integers, $N(\nu)$ the number of ν_n not exceeding ν, $D(\nu)=N(\nu)/\nu$, $D^\cdot=\lim \sup D(\nu)$, $D^\cdot(\nu)=\sup_{x \geqq \nu} D(x)$. Let $\{\lambda_n\}$ be the sequence complementary to $\{\nu_n\}$ and let $D^\cdot < \frac{1}{2}$. Let $F(x)$ be continuous for $x \geqq 0$, with $\log F(x)$ a convex function of $\log x$; let $F(x)$ have

a λ_1th derivative such that $F^{(\lambda_1)}(x) = o\{[F(x)]^{1+1/\lambda_1}\}$; set $\log F(e^\sigma) = p(\sigma)$ and suppose that

$$\int^\infty p(\sigma) \exp\left\{ -\int^\sigma \frac{du}{1 - 2D \cdot (\beta p(u))} \right\} d\sigma = \infty$$

for some $\beta > 0$; let $f(x)$ be a continuous function in $(-\infty, \infty)$ such that $\lim_{|x| \to \infty} f(x)/F(|x|) = 0$. Then for every $\epsilon > 0$ there is a polynomial $P(x) = a_0 + a_1 x^{\lambda_1} + \cdots + a_n x^{\lambda_n}$ such that $|f(x) - P(x)| < \epsilon F(|x|)$, $-\infty < x < \infty$. For $\lambda_n = n$ we have Bernstein's theorem. The author indicates how his theorem can be derived from his results on generalized quasi-analyticity [Ann. Sci. École Norm. Sup. (3) 63, 351–378 (1947); these Rev. 9, 229]. *R. P. Boas, Jr.*

Berman, D. L. **On an interpolation process of Academician S. N. Bernštein.** Doklady Akad. Nauk SSSR (N.S.) **60**, 333–336 (1948). (Russian)

We consider a triangular matrix (1) $\{x_k^{(n)}\}$, $k = 1, \cdots, n$, such that $-1 \leq x_1^{(n)} < \cdots < x_n^{(n)} \leq 1$ $(n = 1, 2, \cdots)$. Write $\Delta_n = \max (x_{k+1}^{(n)} - x_k^{(n)})$ for $k = 1, \cdots, n-1$, and let $\sigma(a, b)$ denote the number of digits in the nth row of the matrix (1) which satisfy the inequality $a \leq x_j^{(n)} \leq b$. Let

$$\omega_n(x) = \prod_{j=1}^{n} (x - x_j^{(n)}), \quad l_j^{(n)}(x) = \omega_n(x)/(x - x_j^{(n)}) \omega_n'(x_j^{(n)})$$

(where $j = 1, \cdots, n$ and $n = 1, 2, \cdots$). Let $f(x)$ be a function defined in the interval $(-1, 1)$. We write

$$L_n(f, x) = \sum_{k=1}^{n} A_k^{(n)} l_k^{(n)}(x),$$

where $A_k^{(n)} = f(x_k^{(n)})$, $k \not\equiv 0 \pmod{2l}$;

$$A_{2li}^{(n)} = \sum_{j=1}^{l} f(x_{2l(i-1)+2j-1}^{(n)}) - \sum_{j=1}^{l-1} f(x_{2l(i-1)+2j}^{(n)}),$$

where l is an arbitrary fixed natural number. This is Bernstein's interpolation process.

The author proves two theorems. (1) Suppose the matrix (1) satisfies the following conditions: (A) $\Delta_n \to 0$ as $n \to \infty$; (B) at x_0 we have, for $x_k^{(n)} < x_{k+1}^{(n)} \leq x_0$, $|l_k^{(n)}(x_0)| \leq |l_{k+1}^{(n)}(x_0)|$ $(n = n_0, n_0+1, \cdots)$ and, for $x_0 \leq x_k^{(n)} < x_{k+1}^{(n)}$, $|l_k^{(n)}(x_0)| \geq |l_{k+1}^{(n)}(x_0)|$ $(n = n_0, n_0+1, \cdots)$; (C) if $\sigma(x_0, x_1^{(n)}) = h$, then

$$|l_j^{(n)}(x_0)| < K(x_0) \phi(h),$$

where $K(x_0)$ is a nonnegative finite number and $\phi(h)$ is an arbitrary function of h which tends to zero as $h \to \infty$. Then Bernstein's interpolation process defined for a function $f(x)$ which is bounded in $(-1, 1)$ and continuous at x_0 converges to $f(x_0)$ at x_0.

(2) If the matrix (1) is a matrix of Jacobi abscissae [cf. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloquium Publ., v. 23, New York, 1939, p. 328; these Rev. 1, 14] with parameters $-1 \leq \alpha < 0$, $-1 \leq \beta < 0$ and if $f(x)$ is continuous in $(-1, 1)$, then the Bernstein interpolation polynomials converge uniformly to $f(x)$ in $(-1, 1)$.

If (1) is a matrix of Legendre abscissae [cf. Szegő, op. cit., p. 377] then the convergence is uniform in every interval $(-1+\epsilon, 1-\epsilon)$, $0 < \epsilon < 1$. *A. C. Offord* (London).

Special Functions

✦**Relton, F. E.** **Applied Bessel Functions.** Blackie & Son Limited, London, 1946. vii+191 pp. 17s.6d.

This book gives an elementary exposition of the theory and application of the Bessel functions. It has its roots in a course of lectures, given by the author, and is addressed in the first place to technicians. No use is made of contour integration or of the complex variable in the analytic sense. After an introductory chapter, treating the ancillary functions (error-function, beta and gamma function) some points of the theory of linear differential equations with variable coefficients (especially of the second order) are treated in chapter II. In chapter III the cylinder functions are introduced as solutions of the two recurrence formulae

$$C_{n-1}(x) + C_{n+1}(x) = (2n/x) C_n(x),$$
$$C_{n-1}(x) - C_{n+1}(x) = 2C_n'(x).$$

The rest of the book gives, in addition to the traditional topics, a great number of applications to several parts of physics and engineering. Special stress is laid upon the demonstration of the existence and the nature of the zeros of the Bessel functions. The book is concluded by a bibliographical note, which contains a list of the principal textbooks of theory and applications. *S. C. van Veen.*

Wilkins, J. Ernest, Jr. **Nicholson's integral for $J_n^2(z) + Y_n^2(z)$.** Bull. Amer. Math. Soc. **54**, 232–234 (1948).

The author gives a rather elementary proof of the well-known result

$$J_n^2(z) + Y_n^2(z) = (8/\pi^2) \int_0^\infty K_0(2z \sinh s) \cosh 2ns \, ds;$$

n arbitrary complex, $\Re(z) > 0$ [cf. Watson, A Treatise on the Theory of Bessel Functions, Cambridge University Press, 1922, pp. 441–444]. It is shown that

$$y(z) = \int_0^\infty K_0(2z \sinh s) \cosh 2ns \, ds$$

is a particular solution of

$$z^2 y''' + 3z y'' + (1 - 4n^2 + 4z^2) y' + 4zy = 0,$$

with the general solution

(1) $y(z) = A J_n^2(z) + B Y_n^2(z) + C J_n(z) Y_n(z)$
$= (2/\pi z)\{A + (B-A) \sin^2 (z - \tfrac{1}{2} n\pi - \tfrac{1}{4}\pi)$
$\qquad + \tfrac{1}{2} C \sin (2z - n\pi - \tfrac{1}{2}\pi) + O(z^{-1})\}.$

The main part of the article is the proof that

$$\lim_{z \to \infty} z y(z) = \lim_{z \to \infty} \int_0^\infty z K_0(2z \sinh s) \cosh 2ns \, ds = \tfrac{1}{4}\pi.$$

[This proof seems to the reviewer to be incomplete, although the result is correct.] This result is incompatible with (1) unless $B = A$, $C = 0$, $A = \tfrac{1}{4}\pi^2$. *S. C. van Veen* (Delft).

Weinstein, Alexander. **Discontinuous integrals and generalized potential theory.** Trans. Amer. Math. Soc. **63**, 342–354 (1948).

By a generalised potential theory in a fictitious space of n dimensions ($n > 0$, n not necessarily integral) the values of the Weber-Schafheitlin discontinuous integrals

(1) $\displaystyle \int_0^\infty J_{q+1}(yt) J_q(bt) dt = \begin{cases} 0, & 0 \leq y < b, \\ (2b)^{-1}, & y = b, \\ b^q y^{-q-1}, & y > b, \end{cases}$

are obtained in a simple way. It is first shown that, for $x > 0$,

$$\psi_b(x, y) = -2^{2q} \Gamma^2(q+1) \Gamma^{-1}(2q+1) b^{-q} y^{q+1}$$
$$\times \int^\infty \sigma^{-x} J_{q+1}(yt) J_q(bt) dt$$

is a solution of the potential equation $y(\psi_{xx} + \psi_{yy}) - p\psi_y = 0$ in a fictitious "space" of $n = p+2 = 3+2q$ dimensions ($n > 2$, $p > 0$, $q > -\tfrac{1}{2}$); $\psi_b(x, y)$ is a regular function of x and y with

the exception of the point $B(0, b)$, which is a branch point of infinite order. The values (1) are special cases $(\beta = 0, \frac{1}{2}\pi, \pi)$ of the general result

$$\lim_{Q \to B} \psi_b(x, y) = -x^{-\frac{1}{2}}\Gamma^{-1}(q+\frac{1}{2})\Gamma(q+1)\beta,$$

where the point $Q(x, y)$ tends to $B(0, b)$ on a smooth curve in such a way that $\lim_{Q \to B} (b-y)/x = \cot \beta$ $(0 \leqq \beta \leqq \pi)$; $x \geqq 0$. As a by-product, explicit formulas are obtained for the fundamental solutions of the generalised Stokes-Beltrami equations $y^p \partial \varphi / \partial x = \partial \psi / \partial x$, $y^p \partial \varphi / \partial y = -\partial \psi / \partial x$.

$\qquad\qquad\qquad\qquad$ *S. C. van Veen* (Delft).

Erdélyi, A. Expansions of Lamé functions into series of Legendre functions. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 247–267 (1948).

The author starts by quoting some previous expansions of Lamé's functions into series of Legendre functions by Darwin and Hobson. The present work starts from the transformation by Ince of Lamé's equation in the form involving elliptic functions. The author quotes some formulas for elliptic functions and Ferrer's definition of associated Legendre functions. Expansions for the latter functions are given, including the addition theorem. A solution of Lamé's differential equation in the form of an infinite series of associated Legendre functions is then given. By substituting this series solution into the differential equation recurrence relations are obtained for the coefficients of the series. The condition for the consistency of the recurrence relations leads to a transcendental equation for the parameter of the differential equation. The author proves that the series solution is convergent within a certain circle in the complex plane of the independent variable. Other series solutions of Lamé's differential equation in terms of associated Legendre functions are derived from integral equations for the solutions. Finally the author obtains series solutions for all the even and odd Lamé functions of the different known forms.

So far this work relates to Lamé functions of integral index; the author then proceeds to discuss transcendental Lamé functions of nonintegral index. Special attention is given to the periodicity of the functions. The conditions leading to Lamé functions of real and of imaginary period are discussed. Finally the author mentions some further expansions of Lamé functions by series of Legendre functions of fractional order. $\quad$ *M. J. O. Strutt* (Zurich).

Bailey, W. N. Identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (2) **50**, 1–10 (1948).

It is shown that much of the structure of identities in basic hypergeometric series including identities of the Rogers-Ramanujan type may be developed directly from a single fundamental principle, namely, if $\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}$ and $\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}$ then $\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n$. In the applications δ_n are selected so that the series for γ_n may be summed. Then by suitable selection of the α_n and β_n from known expansions or identities, results may be obtained as desired. Several generalized expansions are derived to illustrate the method. From these a number of new identities of the Rogers-Ramanujan type are obtained by specialization. $\qquad\qquad$ *N. A. Hall* (Minneapolis, Minn.).

Burchnall, J. L., and Chaundy, T. W. The hypergeometric identities of Cayley, Orr, and Bailey. Proc. London Math. Soc. (2) **50**, 56–74 (1948).

Earlier work of Orr [Trans. Cambridge Philos. Soc. **17**, 1–15 (1899)] is improved and extended to obtain a number of identities involving sums of products of hypergeometric series of various orders. The basic technique depends upon manipulation of the differential equations corresponding to the series, although a number of familiar identities are employed in certain proofs. These results are expressed frequently in terms of double hypergeometric series and as such represent extensions from previous work of the authors [Quart. J. Math., Oxford Ser. **12**, 112–128 (1941); these Rev. **3**, 118]. $\qquad\qquad$ *N. A. Hall* (Minneapolis, Minn.).

Broadbent, D., and Jánossy, L. Production of penetrating particles in extensive air showers. Proc. Roy. Soc. London. Ser. A. **192**, 364–382 (1948).

In this paper the authors use integrals of the type

$$(F_1, F_2, \cdots, F_n) = \gamma \int_0^\infty \prod_{k=1}^{n} (1 - e^{-F_k x}) x^{-\gamma-1} dx,$$

$F_k > 0$, $n > \gamma > 0$ (γ not an integer). No difficulties arise if all the quantities F are equal or nearly so, but for very large values of n the number of terms is large and the evaluation becomes cumbersome. For $n \gg 1$, $(1 - e^{-Fx})^n \approx 1$ or 0 according as $x > (\log n)/F$ or $x < (\log n)/F$, and

$$\gamma \int_0^\infty (1 - e^{-Fx})^n x^{-\gamma-1} dx \sim (\log n)^{-\gamma} F^\gamma.$$

Difficulties arise if the F are greatly different. In the simplest case, $\gamma \int_0^\infty (1 - e^{-\alpha x})^j (1 - e^{-x})^k x^{-\gamma-1} dx$ $(\alpha \ll 1)$ the approximate value

$$\gamma \alpha^\gamma \int_0^\infty (1 - e^{-x})^j \frac{dz}{z^{\gamma+1}} + \gamma(-\alpha)^j \left\{ \frac{\partial^j}{\partial f^j} \int_0^\infty (1 - e^{-fx})^k \frac{dz}{z^{\gamma+1}} \right\}_{k=1},$$

neglecting terms of order $j+1$ in α, is reduced to the integrals described above. The examples used in the text are $j = k = 1$, $\gamma = 1\frac{1}{2}$ and $j = 2$, $k = 1$, $\gamma = 1\frac{1}{2}$. The rather cumbersome exact expression in this case is also given.

$\qquad\qquad\qquad\qquad$ *S. C. van Veen* (Delft).

Colombo, Serge. Sur la fonction $\nu(t, n)$. C. R. Acad. Sci. Paris **226**, 1235–1236 (1948).

For

$$\nu(t, n) = \int_n^\infty (t^s / \Gamma(s+1)) ds$$

and

$$\nu i(t, n) = \int_0^t s^{-1} \nu(s, n) ds$$

the author proves the relation

$$\pi^{\frac{1}{2}} \int_0^t \nu(s, 0) s^{-\frac{1}{2}} ds = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \nu i^{(n)}(t, n+\frac{1}{2})$$

by operational methods. [There appears to be a misprint in the definition of νi and also one in the final result.]

$\qquad\qquad\qquad\qquad$ *A. Erdélyi* (Edinburgh).

Harmonic Functions, Potential Theory

Walsh, J. L. Methods of symmetry and critical points of harmonic functions. Proc. Nat. Acad. Sci. U. S. A. **34**, 267–271 (1948).

In previous papers [e.g., same Proc. **34**, 111–119 (1948); these Rev. **9**, 432] the author determined the location of the critical points of a harmonic function $u(x, y)$ by express-

ing grad u as the force due to certain distributions of matter. In the present paper the author introduces, for the study of the critical points of $u(x, y)$, a method based upon topological considerations involving symmetry. The main theorem proved by this method states that, if $u(x, y)$ is harmonic in a region R cut by the axis of reals and if $u(x, y) > u(x, -y)$ whenever $y > 0$ and both points (x, y) and $(x, -y)$ lie in R, then $u(x, y)$ has no real critical points in R. Together with the conformal mapping of R upon the interior of a circle, this theorem leads to a new proof of the following result of the author [Bull. Amer. Math. Soc. **54**, 196–205 (1948); these Rev. **9**, 432]. Let $u(x, y)$ be harmonic but not identically zero in a region R bounded by the Jordan curve C; let it be continuous on $R+C$, nonnegative on an arc C_0 of C and zero on $C-C_0$. Then all the critical points of $u(x, y)$ lie in the subregion of R bounded by C_0 and the line (non-Euclidean relative to R) joining the end-points of C_0. Other applications are given on this method and of similar topological methods involving consideration of level curves.

M. Marden (Milwaukee, Wis.).

Nicolesco, Miron. **Approssimazione delle funzioni armoniche in più variabili mediante polinomi armonici.** Univ. Roma. Ist. Naz. Alta Mat. Rend. Mat. e Appl. (5) **6**, 410–423 (1947).

The author proves that any function u bounded and harmonic in a closed region D in three dimensions is the uniform limit of a sequence of harmonic polynomials. The theorem, which is a generalization to space of the theorem of Runge, is proved by expressing u by means of Green's formula, replacing the integrals by finite sums, and expanding the resulting special harmonic functions in harmonic polynomials. An extension is also made to three dimensions of a theorem of Lebesgue, namely, that any continuous function in D is the uniform limit of a sequence of functions each of which is the limit of a sequence of harmonic polynomials. The methods may be used to extend either theorem to any number of dimensions greater than one.

J. W. Green (Los Angeles, Calif.).

Radon, I. [J.]. **On boundary-value problems for the logarithmic potential.** Uspehi Matem. Nauk (N.S.) **1**, no. 3–4(13–14), 96–124 (1946). (Russian)

Translated from Akad. Wiss. Wien, S.-B. IIa. **128**, 1123–1167 (1920).

Zaremba, S. **On a mixed problem for Laplace's equation.** Uspehi Matem. Nauk (N.S.) **1**, no. 3–4(13–14), 125–146 (1946). (Russian)

Translated from Bull. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Ser. A. **1910**, 313–344.

Tolotti, Carlo. **Sulla struttura delle funzioni iperarmoniche in piu' variabili indipendenti.** Giorn. Mat. Battaglini (4) **1(77)**, 61–117 (1947).

Let u be an n-hyperharmonic function in m variables, that is, a function satisfying $\Delta^{2n}u = 0$. The problems investigated in this paper are those related to representing u in a domain D by one of the two forms (1) $\sum_{0}^{n-1}x^k u_k$ and $\sum_{0}^{n-1}\rho^{2k}u_k$, where u is harmonic, x is the directed distance from some hyperplane Π, and ρ is the distance from some point Q. It has previously been shown that such representations are possible if D is normal with respect to Π or Q, respectively. In the first case, normality means that every line perpendicular to Π meets D in a segment; in the second, that every half line issuing from Q meets D in a segment.

The author proves that in case $m > 2$, normality of D is also necessary for these representations to be possible. This is not the case when $m = 2$; then representation (1) is possible in every simply connected region. The author also studies the degree of freedom with which the u_k can be chosen when representations (1) and (2) are possible, and investigates several other special representations similar to (1) and (2), but possible in more general regions.   *J. W. Green.*

Differential Equations

*✶Blanc, Charles. **Les Équations Différentielles de la Technique.** Éditions du Griffon, Neuchatel, 1947. 315 pp. 29.50 Swiss francs; bound, 34.50 Swiss francs.

This book is based upon a course in applied mathematics given at the University of Lausanne. It is divided into three parts, the first two of which deal with ordinary differential equations and with partial differential equations. The third part consists of three chapters, devoted to the calculus of variations, elliptic functions and Bessel functions.

In the first two parts attention is confined to linear equations, and indeed mostly to equations having constant coefficients. No more attention is paid to questions of logical rigor than will be appreciated by the better engineering students. However, within these limitations the treatment is surprisingly complete and mature. In dealing both with ordinary and partial differential equations the author organizes the discussion systematically around the following four classes of problems. (1) Problems relating to free oscillations of physical systems. (2) Problems concerning motions subject to prescribed initial conditions. (3) Problems concerning periodic motions in response to periodic driving forces. (4) Boundary value problems. Undoubtedly, the readers for whom the book is intended will find this plan very helpful. On the whole, the reviewer considers these first two parts to be as good an introduction to differential equations, for students of engineering and physics, as he has seen.

The third part of the book seems to be somewhat less satisfactory. Here, also, the exposition is always clear and competent. However, the brevity of the chapters (of 21, 24, and 30 pages, respectively), and the fact that no use is made of analytic function theory, make it inevitable that what is given amounts to only a meager introduction to the large subjects which are under consideration.

The book contains no exercises for the reader, but it does contain many interesting examples which are discussed in full. There is a table of Laplace transforms, and short tables of the elliptic and Bessel functions. Each chapter closes with a useful list of references to other works where the theory is treated more completely. The typography, and the physical appearance of the book generally, are excellent.

L. A. MacColl (New York, N. Y.).

*✶Hopf, L. **Introduction to the Differential Equations of Physics.** Translated by Walter Nef. Dover Publications, New York, N. Y., 1948. v+154 pp. $1.95.

[Translation of Einführung in die Differentialgleichungen der Physik, de Gruyter, Berlin and Leipzig, 1933.] The book begins with a discussion of ordinary differential equations of the mechanics of particles. The equations of motion of projectiles, planets, systems of particles, particles moving under frictional forces and systems of particles and springs

are included. The operators of vector calculus including gradient, divergence and curl, and the principal theorems in that subject, are then introduced. This is followed by brief developments of partial differential equations of physics including the equations of Laplace, Poisson, Maxwell and the wave equation. The use of characteristic functions in solving boundary value problems of physics is illustrated by a few examples. The book ends with very brief discussions of waves, conformal mapping and the use of sources. Some of the examples presented in the book are carried to completion; others are dropped after a brief formulation of the problem, owing to lack of space. The mathematical analysis is limited to formal procedures. There is but one reference to fuller treatments of the subjects covered. No exercises are given. The English translation is accurate and clear. *R. V. Churchill* (Ann Arbor, Mich.).

⊀Kamke, E. **Differentialgleichungen. Lösungsmethoden und Lösungen. Band I. Gewöhnliche Differentialgleichungen.** 2d ed. Mathematik und ihre Anwendungen in Monographien und Lehrbüchern. Band 18. Akademische Verlagsgesellschaft, Leipzig, 1943; J. W. Edwards, Ann Arbor, Mich., 1945. xxvii+642 pp. $14.00.

A third edition appeared in 1944 and was reviewed in these Rev. 9, 33.

Viguier, Gabriel. **Réduction de l'équation de Riccati à l'équation linéaire du second ordre à coefficients constants.** Bol. Soc. Portuguesa Mat. Sér. A. 1, 41–48 (1948).

The author first reduces the general Riccati equation $u'+Pu^2+Qu+R=0$ to an equation of the second order $y''-a_2(x)y=0$, and then shows that this equation may be transformed into an equation $d^2z/dt^2+A_1 dz/dt+A_2z=0$ with constant coefficients by means of the transformation $y=\lambda(x)z$, $dt=u(x)dx$ if and only if $a_2(x)=c_4/(c_2x^2+c_1x+c_0)^2$, $u(x)=(c_2x^2+c_1x+c_0)^{-1}$, $\lambda(x)=(c_2x^2+c_1x+c_0)^{\frac12}$, where c_0, c_1, c_2 and c_4 are constants. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Mitrinovitch, Dragoslav S. **Sur une classe d'équations différentielles du premier ordre que l'on rencontre dans divers problèmes de géométrie.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. 6, 99–120 (1939).

This article contains a number of results connecting the solutions of different types of differential equations. A typical result is the following. If one knows a particular solution of the equation $y'=\{\alpha_0(x)y^2+\alpha_1(x)y+\alpha_2(x)\}/\{y+\beta(x)\}$ then this equation can be transformed into an equation of the form $y'^2+y^3=F(x)$. *J. E. Wilkins, Jr.*

Mitrinovitch, Dragoslav S. **Quelques propositions relatives à l'équation différentielle de Riccati.** Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. 6, 121–156 (1939).

If the Riccati equation $y'+y^2=\phi(x)$ has a solution $y_0(x)$, the author constructs several recursively defined families of functions $\phi_k(x)$ $(k=1,\cdots)$ such that the solution y_k of the equation $y_k'+y_k^2=\phi_k(x)$ can be expressed in terms of y_{k-1} and ϕ_{k-1}, the only operations required being differentiation and integration of ϕ_{k-1} and the rational operations. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Cambi, Enzo. **Trigonometric components of a frequency-modulated wave.** Proc. I. R. E. 36, 42–49 (1948).

The problem is dealt with in a manner avoiding the simplification usually found in the literature, leading to a differential equation of Mathieu's type. The author, instead, arrives at the following basic differential equation:

$$r^2(1+2\gamma\cos x)d^2q/dx^2+q=0.$$

This equation is related to a nondissipative circuit. The solution is given in the form of a complex Fourier series. The coefficients of this series are found from an infinite number of homogeneous equations. The consistency of these homogeneous equations requires the existence of a determinantal equation for the parameters in the differential equation. This determinantal equation differs, however, from the well-known determinant connected with Hill's differential equation. The author proceeds to discuss the solutions in the linear nonhomogeneous case, including the amplitude of the resonances. He obtains approximate expressions in the case where γ is small. By a numerical discussion he exhibits the magnitudes of the amplitudes of the solution. He then discusses the stability of the solution in a number of cases and shows that the number of instable regions is only half the number in the case of Mathieu's equation. Finally some brief remarks are made on the case of the dissipative circuit. *M. J. O. Strutt* (Zurich).

Filippov, A. F. **Sufficient conditions for the uniqueness and nonuniqueness of the solution of a differential equation.** Doklady Akad. Nauk SSSR (N.S.) 60, 549–552 (1948). (Russian)

The new results are the following. (I) Let $f(x, y)$ be continuous for $0\leqq x\leqq a$, $y\leqq b$; let $F(x, y)$ be nonnegative and continuous for $0<x\leqq a$, $0\leqq y\leqq b$, with $F(x, 0)=0$; and let the equation $y'=F(x, y)$ have a solution $u(x)$ with $u(0)=0$, $u'(0)=0$ and $u(a)>0$. If $y'=f(x, y)$ has a solution $v(x)$ through $(0, 0)$ and $f(x, y)-f(x, v(x))\geqq F(x, y-v(x))$ when $0<x\leqq a$ and $v(x)\leqq y\leqq v(x)+u(x)$, then the solution $v(x)$ is not unique. Sufficient conditions for the existence of a solution such as $u(x)$ are given. They are that $F(x, y)=\varphi(y)/\psi(x)$ and either (1) $\varphi(t)\geqq mt$, $m>0$, and

$$\lim_{z\to 0}\int_z^h\left\{\frac{1}{\psi(t)}-\frac{1}{\varphi(t)}\right\}dt=\infty\,;$$

or (2) $\varphi(t)/t\to\infty$ as $t\to 0$ and

$$\liminf_{z\to 0}\int_z^h\left\{\frac{1}{\psi(t)}-\frac{1}{\varphi(t)}\right\}dt>-\infty.$$

(II) Assume that $f(x, y)$ is continuous, that $\varphi(t)$ and $\psi(t)$ are continuous and positive for $t>0$, and in addition that $|f(x, y_1)-f(x, y_2)|\leqq\varphi(|y_1-y_2|)/\psi(x)$ in a neighborhood of $(0, 0)$, $x>0$. Then either of the following conditions implies that $y'=f(x, y)$ has a unique solution through $(0, 0)$, $x>0$:

(i) $$\liminf_{z\to 0}\int_z^h\left\{\frac{1}{\psi(t)}-\frac{1}{\varphi(t)}\right\}dt=-\infty\,;$$

or (ii) $\varphi(t)\leqq Mt$, $M>0$, and

$$\liminf_{z\to 0}\int_z^h\left\{\frac{1}{\psi(t)}-\frac{1}{\varphi(t)}\right\}dt<\infty.$$

J. P. LaSalle (Notre Dame, Ind.).

Stampacchia, Guido. **Sulle condizioni che determinano gli integrali dei sistemi di equazioni differenziali ordinarie del primo ordine.** Giorn. Mat. Battaglini (4) 1(77), 55–60 (1947).

This paper is concerned with sufficient conditions for a system of n first-order ordinary differential equations

$$y_i'=f_i(x; y_1, \cdots, y_n), \quad i=1, \cdots, n; a\leqq x\leqq b,$$

to admit a solution which intersects given varieties S_i lying in respective specified hyperplanes $x=a_i$ ($a \le a_i \le b$; $i=1, \cdots, n$). For brevity, there will not be listed here the hypotheses imposed upon the S_i and the functions $f_i(x; y_1, \cdots, y_n)$. In particular, the results of this paper extend those obtained by the author for the special case $n=2$ in an earlier paper [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2**, 411–418 (1947); these Rev. **9**, 35].   *W. T. Reid* (Evanston, Ill.).

Constantinesco, G. G. Sur les intégrales de certaines classes d'équations différentielles linéaires. Acad. Roum. Bull. Sect. Sci. **24**, 513–516 (1943).

Lorsqu'on envisage une équation différentielle ordinaire linéaire d'ordre n,

$$(1) \qquad p_0(x)y^{(n)}+p_1(x)y^{(n-1)}+\cdots+p_n(x)y=0,$$

dont les coefficients sont des polynomes en x que l'on peut toujours supposer du même degré m et l'on se propose de représenter les solutions à la manière de Laplace,

$$(2) \qquad y(x)=\int_\alpha^\beta V(z)e^{zx}dz,$$

on trouve, avec Poincaré [cf. Amer. J. Math. **7**, 203–258 (1885), p. 217 = Oeuvres, t. 1, pp. 226–289; p. 241], que, si l'intervalle (α, β) satisfait aux conditions

$$(3) \qquad \left\{\frac{d^h}{dz^h}|z^k V(z)|e^{zx}\right\}_\alpha^\beta=0,$$

$$h=0, 1, \cdots, m-1; \; k=0, 1, \cdots, n,$$

la fonction inconnue $V(z)$ doit satisfaire à une équation linéaire d'ordre m dont les coefficients sont des polynomes au plus de degré n (par rapport à z).

En appliquant à cette équation en V la même remarque, on est conduit à construire l'équation en $W(u)$, telle que

$$(2') \qquad V(z)=\int_\lambda^\mu W(u)e^{zu}du$$

si l'intervalle (λ, μ) satisfait aux conditions de Poincaré. L'auteur démontre que, au changement du signe de x près, on retrouve l'équation (1).

Donc, les équations linéaires, dont les coefficients sont des polynomes, sont invariantes par rapport à deux transformations de Laplace ou, si l'on veut, que les solutions des équations (1) satisfont à l'équation fonctionelle

$$y(x)+\int_\alpha^\beta e^{zx}dz\int_\lambda^\mu y(x)e^{-zx}dx=0$$

si les intervalles (α, β), (λ, μ) satisfont aux conditions convenables.   *G. Lampariello* (Messina).

Pipes, Louis A. The analysis of retarded control systems. J. Appl. Phys. **19**, 617–623 (1948).

Hok, Gunnar. Response of linear resonant systems to excitation of a frequency varying linearly with time. J. Appl. Phys. **19**, 242–250, 623 (1948).

The author studies the response of linear passive networks subject to a driving term of the form $E \exp [j(\omega_0 t + \epsilon t^2)]$, where ω_0 and ϵ are constants. The solution of this problem is obtained with the aid of the Laplace transform. Universal curves are presented so that they will be useful in engineering problems. Some applications are indicated.
                    A. E. Heins (Pittsburgh, Pa.).

Volk, I. M. On periodic solutions of autonomic systems. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 29–38 (1948). (Russian)

This paper is a continuation of two previous articles by the author [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 559–574 (1946); same journal **11**, 433–444 (1947); these Rev. **8**, 330; **9**, 185] on the construction of periodic perturbations of differential equations which depend in a nonregular manner on a real parameter. Let $dx_\nu/dt=X_\nu(x_1, \cdots, x_n; \mu)$ ($\nu=1, \cdots, n$) be a system of ordinary differential equations whose right sides are regular analytic functions of the $x_1, \cdots, x_n$ and, at $\mu=0$, meromorphic functions of μ. In contrast to the two previous papers the X_ν are here independent of t. If only the leading terms with respect to μ are preserved in the right members, a simplified differential system is obtained, of which a periodic solution, bounded at $\mu=0$, is assumed to be known. The problem is to construct a periodic solution of the full system that tends to this given solution of the simplified system, as $\mu \to 0$. If certain conditions, too involved to be stated here, are satisfied, such a periodic solution can be found as a convergent series of the form $x_\nu = \sum_{r=0}^\infty y_r(t, \mu)\mu^r$, where the $y_r(t, \mu)$ are periodic solutions of a linear differential system whose homogeneous part is a somewhat modified form of the usual variational system. The $y_r(t, \mu)$ are bounded but not necessarily analytic at $\mu=0$. [It seems to the reviewer that the proof given is complete only for the special case that the variational equations have constant coefficients, since it is based on an incorrect statement in the first of Volk's three papers concerning the boundedness, at $\mu=0$, of the transformation matrix that reduces the variational system to one with constant coefficients.]   *W. Wasow* (Swarthmore, Pa.).

Tihonov, A. On the dependence of the solutions of differential equations on a small parameter. Mat. Sbornik N.S. **22(64)**, 193–204 (1948). (Russian)

In the system $dx_i/dt=f_i(t, x_i, z)$ ($i=1, \cdots, n-1$), $\mu dz/dt=F(t, x_i, z)$, let μ be a small parameter. The solutions of the reduced system obtained for $\mu=0$ lie on the surface $F(t, x_i, z)=0$ in the (t, x_i, z)-space. It is shown that every integral curve with prescribed fixed initial point tends, as $\mu \to 0$, to a limit curve which is composed of segments parallel to the z-axis connected by arcs of integral curves of the reduced system. The particular branches of the surface $F(t, x_i, z)=0$ on which these arcs are situated are characterized in terms of the position of these branches with respect to the vector field defined by the differential system. The results are then generalized to systems of the form $dx_i/dt=f(t, x_i, z_j)$, $\mu_j dz_j/dt=F_j(t, x_i, z_j)$ ($i=1, \cdots, n$; $j=1, \cdots, m$), where all μ_j tend independently to zero. The second part of the paper contains a simple new proof of Poincaré's theorem concerning the analytic dependence on a parameter of solutions of differential equations, if the latter are analytic in this parameter.   *W. Wasow.*

Letov, A. M. The regulation of the stationary state of a system subjected to constant perturbing forces. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 149–156 (1948). (Russian)

A certain mechanical system with a regulator gives rise to the following system:

$$(1) \qquad \begin{aligned} &T^2\ddot\varphi+U\dot\varphi+k\varphi+\mu=\chi, \quad \dot\varphi-\dot\beta-n\beta=-n\epsilon, \\ &\dot\mu+\rho\mu=F(\sigma), \quad \sigma=a\varphi+E\dot\varphi+G^2\ddot\varphi-l^{-1}\mu, \end{aligned}$$

where μ is the displacement of the regulator, σ is the displacement of the starting device, φ and β are the angular displacements of the mechanism from zero (the desired position). In addition T, U, k, n are constants of the basic mechanism; a, E, G, l are constants of the regulator; χ and ϵ are constant perturbing forces; ρ is a constant of the servomotor such that in the first approximation $\rho\mu$ is to compensate for the load on the servomotor. Finally $F(\sigma)$ is a known characteristic function which is continuous and of the same sign as σ. A stationary régime (all derivatives zero) exists whose stability both in the large and in the small is investigated by Liapounoff's second method [A. Liapounoff, Problème Général de la Stabilité du Mouvement, reprinted as Ann. of Math. Studies, no. 17, 1947; these Rev. **9**, 34] and in particular utilizing a theorem of A. I. Lourie [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **9**, 353–367 (1945); these Rev. **7**, 300]. The case where $\rho=0$ (servomotor independent of load) is also discussed by Liapounoff's first method. Self-oscillations are treated following a linear approximation method due to B. V. Bulgakov [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 313–332 (1946); these Rev. **8**, 207] which consists in replacing $F(\sigma)$ by a linear approximation $h\sigma$ and looking for periodic solutions of the resulting linear system. Necessary and sufficient conditions for stability are discussed for the two cases of hard back coupling [$G=0$, $l\neq\infty$] and no back coupling [$G\neq 0$, $l=\infty$].

S. Lefschetz (Princeton, N. J.).

Erugin, N. P. On asymptotically stable solutions of certain systems of differential equations. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 157–164 (1948). (Russian)

Shtokalo [Rec. Math. [Mat. Sbornik] N.S. **19(61)**, 263–283 (1946); these Rev. **8**, 329] considered the stability of solutions of differential equations of the type (1) $dy/dt=(A+\epsilon F(t))y$, where A is a constant matrix, ϵ a small parameter and $F(t)$ is an almost periodic matrix whose elements have the form $\sum c_n e^{i\alpha_n t}$. The author combines part of the analysis of Shtokalo's paper with a modification of a method of Liapounoff due to himself to treat the nonlinear equation (2) $ds/dt=(A+\epsilon F(t))s+G(s)$, where $G(s)$ is a vector whose elements are nonlinear functions of the elements of s. R. Bellman (Stanford University, Calif.).

Haag, Jules. Sur les oscillateurs à amplitude stabilisée. C. R. Acad. Sci. Paris **226**, 1567–1568 (1948).

The author considers a nonlinear oscillating system proposed by Abelé [C. R. Acad. Sci. Paris **225**, 1270–1271 (1947); these Rev. **9**, 239]. The differential equation is $\ddot{x}+\omega^2 x=-2\omega\dot{x}g(y)F(x,\dot{x})$, where $y=(x^2+\dot{x}^2\omega^{-2})^{\frac{1}{2}}$ and F and g satisfy certain inequalities, with $g(a)=0$. It is shown that the solution $x=a\cos\omega t$ is stable and is a limit cycle. It is shown that the period would be sensitive to the introduction of friction; however, if F depends only on y and the frictional term is simply $k\dot{x}$ ($k=$constant), only the amplitude is affected. W. Kaplan (Ann Arbor, Mich.).

Graffi, Dario. Sopra alcune equazioni differenziali non lineari della fisica-matematica. Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) **7**, 121–129 (1940).

The author discusses the nonlinear equation

$$y''+f(y,y')y'+g(y)=0$$

associated with electronic oscillators. Under various conditions on $f(y,y')$ and $g(y)$, which are not necessarily polynomials, the oscillatory nature of any integral, the amplitude

of maximum and minimum values and the existence of periodic solutions are treated. It is assumed that one and only one solution exists with prescribed values of y and y' at $t=t_1$, for any t_1, and that this solution may be continued over the infinite t-interval. The existence of periodic solutions is demonstrated in an elementary manner, i.e., without the use of topology, using a device due to É. and H. Cartan [Annales des Postes, Télégraphes et Téléphones **14**, 1196–1207 (1925)]. R. Bellman (Stanford University, Calif.).

Hartman, Philip. Unrestricted solution fields of almost-separable differential equations. Trans. Amer. Math. Soc. **63**, 560–580 (1948).

The author considers in the first part of the paper the differential equation (1) $dx/dt=g(x)+f(t)$, where $g(x)$ and $f(t)$ are continuous over $-\infty<x<\infty$ and $0\leq t<\infty$, respectively. It is also assumed that $g(x)$ is negative or positive according as x is positive or negative, that $|g(x)|\to\infty$ as $x\to\pm\infty$, and that $f(t)\to 0$ as $t\to+\infty$. It is then shown that as t increases no continuation of any solution of (1) can cease to exist at a finite value of t, and that $x\to 0$ as $t\to+\infty$ holds for every continuation of every solution of (1). Various related results are derived under weakened hypotheses.

The second part of the paper is devoted to a discussion of the asymptotic properties of the solutions of (2) $d^2y/dt^2=(1+f(t))y$, where $f(t)\to 0$ as $t\to+\infty$. This topic is related to the first by means of the substitution $u=(dy/dt)/y$ which transforms (2) into (3) $du/dt=1-u^2+f(t)$. With $u-1=x$, (3) is an equation of type (1). It is then demonstrated that for any solution of (2), the limit

$$(4) \qquad \lim_{t\to+\infty} y\exp\left(-t-\tfrac{1}{2}\int_0^t f(s)ds\right)$$

exists. This relation is sharpened to an asymptotic equality if it is further assumed that $\int_0^\infty |f(t)|^p dt<\infty$ for some p, $1\leq p\leq 2$. R. Bellman (Stanford University, Calif.).

Wintner, Aurel. A norm criterion for non-oscillatory differential equations. Quart. Appl. Math. **6**, 183–185 (1948).

Set (1) $D(x)=x''+f(t)x$, $a^*=a(t)\int_t^\infty du/a^2(u)$, and agree, following Kneser, to call the differential equation $D(x)=0$ oscillatory or non-oscillatory according as each or none of its solutions $x(t)\not\equiv 0$ has zeros clustering at $t=\infty$. The author proves that the differential equation $D(x)=0$ is of non-oscillatory type if and only if there exists some positive function $a(t)$ for which $D(a)$ and a^* are continuous and satisfy the inequality (2) $\int^\infty a^*|D(a)|dt<\infty$. Applications of the result are given. R. Bellman.

Wintner, Aurel. On the counting of nodal curves and surfaces. J. Chem. Phys. **16**, 405–406 (1948).

In the theory of the m-dimensional wave equation $\Delta\Phi+f\Phi=0$ ($m=1, 2, 3$) the cases with radial symmetry are in the foreground. By the method of separation of variables they allow the wave equation for $m=3$ (or $m=2$) to be reduced to $m=1$, namely, to $\Delta_1\rho+f_1\rho=0$, where $\Delta_1=d^2/dr^2$ is the one-dimensional div grad, f_1 is a function of r alone and ρ denotes the r-dependent factor of Φ. The counting of nodal curves and surfaces leads to the determination of the zeros of the normalized solution $\rho=\rho(r)$ of the ordinary differential equation $\rho''+f_1(r)\rho=0$ by Sturm's comparison or separation theorem. The classical procedure fails if the potential is not a function of r alone (or at any rate if the wave equation is not separable). It is the purpose of the present paper to state that, when formulated in an appro-

priate manner, Sturm's theorem can be extended to the case of partial differential equations. In a Euclidean space (or plane), let D and T be two domains, the first of which is bounded and the second connected, and let $T+S$ be contained in D, if S denotes the boundary of the open set T. Then the author gets the multidimensional generalization of Sturm's theorems as follows: every solution $u \neq 0$ of $\Delta u + fu = 0$ must change sign in T, if there exists on $T+S$ a continuous g, satisfying $f \geq g$, and having the property that $\Delta v + gv = 0$ has on $T+S$ a solution v which vanishes on S and does not change sign on T. Some special cases are given for the practical applicability of the lemma.

M. Pinl (Cologne).

Nath Sharma, Prithvi. On some vibrational problems. Math. Student **14** (1946), 63–64 (1948).

Christopherson [Quart. J. Math., Oxford Ser. **11**, 63–65 (1940); these Rev. **2**, 30] and Sen have provided incomplete solutions for the equation $z_{xx} + z_{yy} + b^2 z = 0$ subject to the boundary conditions z (or $\partial z/\partial n$) $=0$ on the boundary of an equilateral triangle. The author points out that Seth [Proc. Indian Acad. Sci., Sect. A. **12**, 487–490 (1940); these Rev. **3**, 123] has given the complete solution.   *A. E. Heins.*

Eisenhart, L. P. Enumeration of potentials for which one-particle Schroedinger equations are separable. Physical Rev. (2) **74**, 87–89 (1948).

The solution of the scalar Helmholtz equation $\nabla^2 \psi + k^2 \psi = 0$ is separable for eleven coordinate systems. The Schroedinger equation $\nabla^2 \psi + (k^2 - V)\psi = 0$ will be separable for these coordinate systems only if V satisfies the condition $V = \sum f(X_i)/h_i^2$, where X_i are the generalized coordinates, f is an arbitrary function and h_i is the scale factor corresponding to X_i. In the present paper the author tabulates the possible forms of V for each of the eleven coordinate systems mentioned earlier.   *H. Feshbach* (Cambridge, Mass.).

Wallace, P. R., and LeCaine, J. Elementary approximations in the theory of neutron diffusion. National Research Council of Canada. Division of Atomic Energy. Document no. 1480, i+172 pp. (47 plates) (1946).

This monograph is essentially an exceptionally complete compilation of solutions of various boundary value problems which occur in the context of the following equations.

(I) Equation governing the thermal diffusion of neutrons: (1) $\nabla^2 \rho - (\rho/L^2) = -(\tau/L^2)q(\mathbf{r})$, where $\rho = \rho(\mathbf{r}, t)$ denotes the density of thermal neutrons at time t, τ the mean life of the neutrons before capture, L is the "diffusion length" and $q(\mathbf{r})$ is the source density. Solutions of this equation are considered in infinite and semi-infinite media with laminar and point sources; and also in media in the form of plane-parallel slabs, spheres, spheres with cavities and spherical shells.

A typical example considered is the case of a finite spherical medium with concentric spherical cavity and point source at the center. The corresponding boundary conditions are $\rho = 0$ at $r = a$ (radius of medium) and $-(\tau/L^2)\partial \rho/\partial r = Q/4\pi r^2$ at $r = a$ (radius of cavity) (Q is a constant). The solution is

$$(2) \qquad \rho = \frac{Q\tau \sinh(a-r)/L}{4\pi\{a_1 L \cosh(a-a_1)/L + L^2 \sinh(a-a_1)/L\}r}.$$

Similarly for a finite spherical medium of radius a and a

general spherically symmetric source distribution

$$(3) \quad \rho = \frac{\tau}{Lr \sinh a/L}\Bigg\{\sinh\{(a-r)/L\}\int_0^r r'q(r')\sinh(r'/L)\,dr'$$
$$+\sinh(r/L)\int_r^a r'q(r')\sinh((a-r')/L)dr'\Bigg\}.$$

Equation (1) also governs the conditions in a pile when solutions are considered, for example, which satisfy the boundary conditions $\rho = 0$ at $x = 0$ and a_1 and $y = 0$ and a_2 and $z = 0$ and a_3. In such cases, solutions are sought in the form of a triple Fourier series and the individual Fourier components are considered separately.

(II) Equation governing the slowing down of neutrons: (4) $\partial \chi/\partial \theta = \nabla^2 \chi + s\delta(\theta)$, where $\chi(r, t)$ is the number of neutrons passing a given energy mark per unit time and θ is the so-called "symbolic age" of the neutrons. The term in $s\delta(\theta)$ on the right hand side is equivalent to the boundary condition that $\chi = s$ for $\theta = 0$. If we multiply equation (4) by $e^{-\theta/L^2}$ and integrate over θ from 0 to ∞, we recover equation (1) with s replacing $\tau q/L^2$. A typical example considered is a finite spherical medium with a point source at the center. The corresponding boundary conditions are: $\chi = 0$ at $r = a$, χ finite at $r = 0$. Also $s = S\delta(r)/4\pi r^2$, where $\delta(r)$ is Dirac's δ-function. The solution is

$$(5) \qquad \chi(\theta, r) = (S/2a^2 r)\sum_{m=1}^{\infty} m \sin(m\pi r/a)e^{-m^2\pi^2\theta/a^2}.$$

(III) Equation governing diffusion of thermal neutrons with sources determined by slowing down theory. In this case equations (1) and (4) are considered together, $q(\mathbf{r})$ in (1) being the solution $\chi(\mathbf{r}, \theta_0)$ of (4), θ_0 being a particular value of θ and denoting the symbolic age for thermal neutrons. Thus for the case of a finite spherical medium with a point source of fast neutrons, we must substitute, for q in (3), $\chi(r, \theta)$ given by (5). We thus find

$$(6) \qquad \rho = (S\tau/2r)\sum_{m=1}^{\infty}\frac{m}{a^2 + m^2\pi^2 L^2}e^{-m^2\pi^2\theta_0/a^2}\sin(m\pi r/a).$$

(IV) Problems with multiplication in a single medium. When multiplication occurs (4) is replaced by

$$(7) \qquad \partial \chi/\partial \theta = \nabla^2 \chi + (s + \rho k/\tau)\delta(\theta),$$

where k is the so-called multiplication factor. The solution $\chi(r, \theta_0)$ of equation (7) is then considered as providing the source for thermal neutrons [equation (1)]. In the case of an infinite medium with a plane source (i.e., $s = S\delta(z)$) of fast neutrons the equations to be considered are

$$(8) \qquad \partial^2 \rho/\partial z^2 - \rho/L^2 = -\chi(\theta_0)\tau/L^2$$

and

$$\partial \chi/\partial \theta = \partial^2 \chi/\partial z^2 + (s + \rho k/\tau)\delta(\theta).$$

Letting ϕ, ψ and σ be the Fourier transforms with respect to z of ρ, χ and s, respectively, we have

$$(9) \qquad \zeta^2 \phi + \phi/L^2 = (\tau/L^2)\psi(\theta_0)$$

and

$$\partial \psi/\partial \theta = -\zeta^2 \psi + (\sigma + k\phi/\tau)\delta(\theta),$$

where ζ is the new variable introduced in the Fourier transformation. The solution of equations (9) is

$$\phi = \sigma \tau/\{(1 + \zeta^2 L^2)e^{\zeta^2 \theta_0} - k\}.$$

Inverting this relation, we get ρ. Other problems can be solved similarly.

The "critical size" of a pile is determined by considering the case $s=0$ in equation (7). Thus, for a slab where ρ and χ have to vanish at $z=0$ and $z=2a$, ρ and χ must be given by [cf. equation (5)]

$$\rho = \sum_{n=1}^{\infty} F_n \sin(n\pi z/2a); \quad \chi = \sum_{n=1}^{\infty} g_n e^{-n^2\pi^2 b_0/4a^2} \sin(n\pi z/2a).$$

Substituting these in the equation (1) and (7) (with $s=0$) we get

$$(\tfrac{1}{4}n^2\pi^2 a^{-2}+L^{-2})F_n = \tau L^{-2}e^{-n^2\pi^2 b_0/4a^2} g_n; \quad g_n = kF_n/\tau.$$

Hence

$$k = (1+n^2\pi^2 L^2/4a^2)e^{n^2\pi^2 b_0/4a^2}.$$

This equation for $n=1$ determines the critical value of the thickness $2a$. The solutions for various cases of physical interest are illustrated by numerous graphs.

                                        S. Chandrasekhar (Williams Bay, Wis.).

Naef, R. A. Wärmeleitung im Zylinder. Schweiz. Arch. Angew. Wiss. Tech. **14**, 156–157 (1948).

The author writes down some particular solutions of the heat equation $u_t = a(r^{-1}u_r + u_{rr} + u_{zz})$.   *F. G. Dressel.*

Crank, J., and Godson, S. M. A diffusion problem in which the amount of diffusing substance is finite. III. Diffusion with nonlinear adsorption into a composite circular cylinder. Philos. Mag. (7) **38**, 794–801 (1947).

[For references to parts I and II cf. the following review.] A long porous cylinder is placed in a liquid bath having a finite cross sectional area. The cylinder is composite, consisting of a cylindrical core $0 \leq r \leq b$ encased by a cylindrical shell $b \leq r \leq a$, the two parts having different coefficients of diffusion. Initially the liquid in the bath contains a known uniform concentration of dye substance and the cylinder is free from the dye solution. As the solution diffuses into the cylinder the concentration of dye in the bath itself is assumed to remain uniform throughout the bath; of course it decreases as time t increases. The concentrations of dye in solution within the outer and inner parts of the cylinder are the unknown functions $C_a(r, t)$ and $C_b(r, t)$. The functions $S_a(r, t)$ and $S_b(r, t)$ represent the concentrations of dye deposited in the capillaries of the cylinder. They are known experimentally as functions of C_a and C_b, and taken in this paper to be proportional to the square roots of C_a and C_b, respectively. As a consequence the problem of determining the latter functions is nonlinear. The boundary value problem in the partial differential equations of diffusion is set up. This is approximated by a problem in difference equations which is solved numerically for various values of the dimensionless parameters that are involved. Graphs are given to show the variation of the total mass of dye in a unit length of the cylinder with time, and the variation of concentrations with r.   *R. V. Churchill* (Ann Arbor, Mich.).

Crank, J. A diffusion problem in which the amount of diffusing substance is finite. IV. Solutions for small values of the time. Philos. Mag. (7) **39**, 362–376 (1948).

[Cf. the preceding review.] The solutions of the problems presented in parts I and II of this series [A. H. Wilson, same vol., 48–58 (1948); J. Crank, same vol., 140–149 (1948); these Rev. **9**, 439] are presented here in forms especially suitable for computation for small values of time t. Those problems deal with diffusion of dye material into uniform slabs and solid cylinders, allowing for the deposit of dye into the capillaries from the dye in the diffused

solution. In linear cases, where the ratio of the concentration of deposited material to the concentration of material in solution is constant, the Laplace transformation method is used to get solutions in terms of error functions. The nonlinear problems are subjected to a transformation of variables and replaced by difference equations which can be solved numerically for small t. Tables of numerical results are given showing the total amount of dye in the slabs and cylinders at various times.   *R. V. Churchill.*

Bureau, Florent. Sur l'intégration des équations de propagation des ondes lumineuses dans les milieux cristallins uniaxes. C. R. Acad. Sci. Paris **226**, 1331–1333 (1948).

The author considers the system of differential equations

(1)
$$\frac{\partial^2 u_1}{\partial t^2} - a^2 \Delta u_1 - (a^2 - c^2)\frac{\partial}{\partial y}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) = g_1(x, y, z, t),$$

$$\frac{\partial^2 u_2}{\partial t^2} - a^2 \Delta u_2 + (a^2 - c^2)\frac{\partial}{\partial x}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) = g_2(x, y, z, t),$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. He shows how the "elementary solution" of this system can be obtained immediately from that of the equation

(2)
$$\left\{\frac{\partial^2}{\partial t^2} - a^2 \Delta\right\}\left\{\frac{\partial^2}{\partial t^2} - a^2 \Delta + (a^2 - c^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\}u = 0.$$

The elementary solution of (2) had been obtained previously by the author [Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 379–402 (1947); same C. R. **225**, 402–403 (1947); these Rev. **9**, 356, 188]. This permits the solution of the Cauchy problem for the system (1).   *F. John* (New York, N. Y.).

Difference Equations

Romanovskiĭ, V. A new method for the solution of a homogeneous difference equation with constant coefficients. Doklady Akad. Nauk SSSR (N.S.) **60**, 1317–1320 (1948). (Russian)

The difference equation $\phi_{n+k} - a_1\phi_{n+k-1} - \cdots - a_n\phi_k = 0$ may be written in symbolic form $|E\phi - A|\phi^k = 0$, where E is the square n-rowed unit matrix and

$$A = \begin{Vmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & & \\ a_{n-1} & 0 & 0 & \cdots & 1 \\ a_n & 0 & 0 & \cdots & 0 \end{Vmatrix}.$$

Let C denote the one-rowed matrix $C = \|\phi_n^0, \phi_{n-1}^0, \cdots, \phi_1^0\|$, where the ϕ_i^0 are the initial given values of ϕ. Then $\phi_k = (CA^{k-1})$, in which (CA^{k-1}) denotes the sum of the elements of the matrix CA^{k-1}. Denote the distinct roots of $|E\lambda - A| = 0$ by $\lambda_1, \cdots, \lambda_\mu$, with orders $n_1, \cdots, n_\mu$. The author then shows that the solution of the difference equation may be written

$$\phi_k = \sum_{i=1}^{\mu}(1/(n_i-1)!)D_\lambda^{n_i-1}[\lambda^{k-1}R_n(\lambda)/a_i(\lambda)]_{\lambda=\lambda_i},$$

in which $a_i(\lambda) = |E\lambda - A|(\lambda - \lambda_i)^{-n_i}$ and

$$R_n(\lambda) = \sum_{\alpha=1}^{n}\phi_{n+1-\alpha}^0 A_{\alpha n}(\lambda).$$

                                        W. E. Milne (Corvallis, Ore.).

Wright, E. M. The linear difference-differential equation with asymptotically constant coefficients. Amer. J. Math. 70, 221–238 (1948).

The author considers the general linear difference-differential equation

(1)　$\sum\limits_{m,n=0}^{M,N} A_{mn}(x)y^{(n)}(x+b_m)=v(x), \qquad 0=b_0<b_1<\cdots<b_m,$

where the coefficients $A_{mn}(x)$ are bounded and integrable in any finite interval and approach constants a_{mn} as $x\to+\infty$. The theory of the behavior of solutions of differential equations of corresponding type was initiated by Poincaré and Dini independently, extended to difference equations by Perron and others, and further extended to difference equations with noncommensurable spans by Bochner. References are given in the paper.

For differential-difference equations, Fourier analysis is used in place of the simpler techniques available for differential or difference equations. The author introduces (2) $\|y-z\|=\int_{x_0}^{\infty}|y-z|^2 e^{-\alpha x}dx$ as a measure of the difference between the functions y and z, and estimates $\|y-z\|$ for y a solution of (1) and z a solution of the corresponding equation with $A_{mn}(x)$ replaced by a_{mn}. Under certain assumptions, it is shown that for every y there exists a z for which $\|y-z\|\to0$ as $x_0\to+\infty$, and even a stronger result is obtained. Results are also given connecting $\|y\|$ with $\|v\|$.

　　　　　　　　　R. Bellman (Stanford University, Calif.).

Integral Equations

~~Zeković~~
~~Nikolić~~, I. A. On the Fredholm series. Doklady Akad. Nauk SSSR (N.S.) 59, 423–425 (1948). (Russian)

S. Michlin has given [C. R. (Doklady) Acad. Sci. URSS (N.S.) 42, 373–376 (1944); these Rev. 6, 271] a proof of the convergence of the Fredholm series for an L^2 kernel $k(s,t)$, based on the idea of writing $k(s,t)=k_1(s,t)+k_2(s,t)$, where $k_1(s,t)$ is of finite rank and $k_2(s,t)$ is of small norm. In the present paper the author uses a similar method to prove the same result for kernels satisfying the conditions

$$\|K\|=\left\{\int_a^b ds\left(\int_a^b|k(s,t)|^q dt\right)^{p/q}\right\}^{1/p}<\infty,$$
$$\left\{\int_a^b dt\left(\int_a^b|k(s,t)|^p ds\right)^{q/p}\right\}^{1/q}<\infty,$$

where $p>1$ and $q=p/(p-1)$, the conjugate index to p.
　　　　　　　　　F. Smithies (Cambridge, England).

Chang, Shih-Hsun. A generalization of a theorem of Lalesco. J. London Math. Soc. 22 (1947), 185–189 (1948).

If A and B are L^2 kernels, (1) $K(x,y)=\int_a^b A(x,s)B(s,y)ds$ and $\{\lambda_k\}$ is the set of the Schmidt eigen-values for $K(x,y)$, then the series $\sum|\lambda_k|^{-1}$ is convergent. The decomposition (1) of K is canonical if the right Schmidt eigen-functions of A are identical with the left Schmidt eigen-functions of B; a necessary and sufficient condition for the L^2 kernel K to have a canonical decomposition into n factors is that $\sum|\lambda_k|^{-2/n}<\infty$. 　　　　*C. Miranda* (Naples).

Dressel, F. G. Solutions of bounded variation of the Volterra-Stieltjes integral equation. Univ. Nac. Tucumán. Revista A. 6, 161–166 (1947).

It is shown that the Stieltjes integral equation

$$f(x)=g(x)+\lambda\int_0^x K(x,y)df(y),\qquad 0\le y\le x\le1,$$

the integral being taken in the Young-Stieltjes sense, has a unique solution $f(x)$ of bounded variation if $g(x)$ is of bounded variation and if $K(x,y)$ satisfies the conditions: (a) $K(x,y)$ is Borel measurable in y for every x, (b) $K(x+0,y)=K(x,y)$, (c) $K(x,x)=0$, (d) there exists a bounded monotonic nondecreasing function $T(x)$ such that $(K(x_2,y)-K(x_1,y))\le|T(x_2)-T(x_1)|$. The proof involves the use of iterated kernels. The result is applied to kernels of the form $K(x,y)=\int_y^x G(t,y)(t-y)^{-\alpha}dt$, $\alpha<1$.
　　　　　　　　　T. H. Hildebrandt (Ann Arbor, Mich.).

Trjitzinsky, W. J. Singular integral equations of the first kind and those related to permutability and iteration. J. Math. Pures Appl. (9) 26 (1947), 283–351 (1948).

The author studies the integral equation of the first kind with nonsymmetric kernel of the type introduced by Carleman [Sur les Équations Intégrales Singulières à Noyau Réel et Symétrique, Uppsala, 1923]. For this purpose, parts of Carleman's theory are modified to deal with nonsymmetric kernels. Next, the "permutability problem" $\int_0^1 p(x,t)q(t,y)dt=\int_0^1 q(x,t)p(t,y)dy$ (p given, q unknown) and the equation $f(x,y)=\int_0^1 q(x,t)q(y,t)dt$ (f given positive definite and symmetric, q unknown and possibly nonsymmetric) are studied. Here $p(x,y)$, $q(x,y)$ and $f(x,y)$ are assumed to be L^2 in x and in y, but not necessarily in (x,y). The use of singular values and singular functions is, however, made possible by a process of "regularising" a given function $f(x,y)$, i.e., by finding $a(x)$, $b(y)$ such that $f(x,y)/a(x)b(y)$ is L^2 in (x,y). Finally, the problem of finding a symmetric kernel whose nth iterate is equal to an assigned symmetric kernel is treated on the basis of Carleman's spectral theory. Space does not permit of a statement of any of the results obtained by the author.
　　　　　　　　　G. E. H. Reuter (Manchester).

Gremyačenskiĭ, A. P. On the characteristic values of systems of nonlinear integral equations. Doklady Akad. Nauk SSSR (N.S.) 60, 337–340 (1948). (Russian)

It is proved that there exists a set of real numbers $\mu_1,\cdots,\mu_n$ so that the system $\mu_i u_i(x)=\int K_i(x,y)f_i(y,u_1(y),\cdots,u_n(y))dy$ ($i=1,\cdots,n$; integration over a bounded domain B in a space of a finite number of dimensions) has solutions not identically zero. The hypotheses are: the $K_i(x,y)$ are real, continuous, symmetric, positive definite; the $f_i(y,u_1,\cdots,u_n)$ are real, continuous for y in B and $a_i\le u\le b_i$ ($a_i<0<b_i$); $\partial F/\partial u_i=f_i(y,u_1,\cdots,u_n)$ for some F; $f_i(y,0,\cdots,0)=0$. The result is established by a method of the type used by Golomb [Math. Z. 39, 45–75 (1934)].
　　　　　　　　　W. J. Trjitzinsky (Urbana, Ill.).

Mitrinovitch, Dragoslav S. Correspondance entre l'équation différentielle du second ordre et une équation intégrale de Volterra. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. no. 7, 191–195 (1941).

Let $\phi(x)$ be a solution of the Volterra integral equation $f(x)=\phi(x)-\int_0^x[a(x)-a(y)]\phi(y)dy$ in which $a(x)$ and $f(x)$ are known. If $a'(0)\ne0$ and $z(x)=\phi(x)-f(x)$, the author shows that $f(x)=(z'/a')'-z$, $z(0)=z'(0)=0$, and that as a

consequence of previous work of his on the Riccati equation [same Bull. no. 6, 121–156 (1939); these Rev. 9, 587] there is an infinite sequence of functions $a(x)$ for which the integral equation may be solved explicitly.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Sarmanov, O. V. **On the order of magnitude of a line of regression. II.** Doklady Akad. Nauk SSSR (N.S.) **60**, 545–548 (1948). (Russian)

The following two theorems generalize previous results of the author [same Doklady (N.S.) **59**, 1061–1064 (1948); these Rev. **9**, 442]. There exists no symmetric correlation which satisfies the condition (1) of the review of the earlier paper, and which is such that the lines of regression $\phi(x)$, of y on x, satisfy the condition $|\phi(x)| \geq \lambda(|x| - A)$ for $|x| \geq A$, $\lambda > 1$, $A > 0$. There exists no nonsymmetric correlation satisfying the same condition (1) and which is such that the lines of regression $\phi(x)$, of y on x, and $\psi(y)$, of x on y, satisfy the conditions $|\phi(x)| \geq \tau(|x|)$ for $|x| \geq A > 0$; $|\psi(y)| \geq \tau^{-1}[(1+\mu)|y|]$ for $|y| \geq B > 0$, $\mu > 0$, where $\tau(|x|)$ is a monotone increasing, continuous convex function of $|x|$ such that $\tau(0) = 0$, and $\tau^{-1}(|y|)$ is its inverse function.

H. P. Thielman (Ames, Iowa).

Parodi, Maurice. **Sur les solutions fondamentales d'un type d'équations intégrales singulières.** C. R. Acad. Sci. Paris **226**, 1237–1239 (1948).

Let $Tf(x) = \int_0^\infty K(y, x) f(y) dy$. The author investigates the integral equation $f + \lambda Tf = g$ by means of the operational calculus. He solves the equation formally when the kernel has period 3, that is, when $T^3 f = \epsilon f$ ($\epsilon = \pm 1$) for all f, shows that the only characteristic values are the roots of the equation $\lambda^3 \epsilon + 1 = 0$, and obtains the characteristic functions.

A. Erdélyi (Edinburgh).

Davison, B. **Influence of a large black cylinder upon the neutron density in an infinite non-capturing medium.** National Research Council of Canada. Division of Atomic Energy. Document no. **MT-135** (N.R.C. 1554), i+39+20 pp. (1945).

The author considers the integral equation

$$(*) \quad n(r) = \frac{2}{\pi} \int_a^\infty dr' r' n(r')$$

$$\times \int_{|r-r'|}^{(r^2-a^2)^{\frac{1}{2}}+(r'^2-a^2)^{\frac{1}{2}}} \frac{d\rho}{[\rho^2-(r-r')^2]^{\frac{1}{2}}[(r+r')^2-\rho^2]^{\frac{1}{2}}}$$

$$\times \int_\rho^\infty dt K_0(t),$$

governing the density of neutrons in a uniform noncapturing medium surrounding an infinitely long black cylinder of radius a. In (*) $K_0(t)$ is the Bessel function of order zero, of the second kind, for a purely imaginary argument. For $r \to \infty$, it is known that

$$n(r) = C(a)[a \log (r/a) + \lambda + O(e^{-(r-a)})],$$

where $C(a)$ is a constant depending only on a and λ is the so-called linear extrapolation length. In this paper it is shown that

$$\lambda = 0.7104 + 0.2524a^{-1} + 0.0949a^{-2} - 0.078125a^{-3} \log a$$
$$- 0.0256a^{-3} + O(a^{-4} \log^2 a).$$

The method by which this formula is derived is as follows. In zero approximation and not too far from the surface of the cylinder, $n(r)$ should be roughly the same as in a half-space bounded by a vacuum. In more accurate approximations, but still not too far from the cylinder, the effects of curvature can be treated as small perturbations. However, for distances comparable with the radius of the cylinder this approach becomes invalid as all the successive corrections are of the same order of magnitude; but the author shows (and this is the essential point of the method) that the leading term, the next largest term, etc., in all the successive corrections, can be determined simultaneously. In this manner an expansion valid for all distances from the center is found.

S. Chandrasekhar.

Drăganu, Mircea. **Sur une équation intégrale régissant le phénomène de la diffusion des neutrons.** C. R. Acad. Sci. Paris **226**, 1698–1699 (1948).

It is indicated how the integral equations governing the diffusion of neutrons in plane-parallel and spherical media can be reduced to equations of Fredholm's type.

S. Chandrasekhar (Williams Bay, Wis.).

Holte, Gunnar. **On the space energy distribution of neutrons in a moderator of infinite size.** Ark. Mat. Astr. Fys. **35A**, no. 2, 12 pp. (1948).

This paper considers the space distribution of neutrons in a uniform noncapturing medium with a point source of neutrons on the assumption that scattering of neutrons by an atom is isotropic in the center of gravity system of the neutron and the atom. The present investigation differs from an earlier one by I. Waller [same Ark. **34A**, no. 3 (1947); these Rev. **8**, 587] only in that allowance is made for a dependence of the mean free path λ on energy E of the form $\lambda(E) = \lambda_0 + \lambda_1 (E/E_0)^b$ (λ_0, λ_1, E_0 and b are constants). Solutions are found in an approximation in which the first three terms in the expansion of the angular distribution in spherical harmonics are included. Expressions for the moments $\overline{r^2}$ and $\overline{r^4}$ of the distribution are also found.

S. Chandrasekhar (Williams Bay, Wis.).

Chandrasekhar, S. **On the radiative equilibrium of a stellar atmosphere. XXII.** Astrophys. J. **107**, 48–72, 188–215 (1948).

In paper XXI of this series [same J. **106**, 152–216 (1947); these Rev. **9**, 444] the author found approximate solutions for problems of diffuse reflection and transmission of radiation by a plane parallel atmosphere of finite optical thickness τ_1. In the present paper it is shown that the exact solutions of these problems can be obtained by solving a pair of functional equations, corresponding to the single functional equation of paper XIV [same J. **105**, 164–203 (1947); these Rev. **8**, 467]. The equations are

$$X(\mu) = 1 + \mu \int_0^1 \frac{\Psi(\mu')}{\mu + \mu'} [X(\mu) X(\mu') - Y(\mu) Y(\mu')] d\mu',$$

$$Y(\mu) = e^{-\tau_1/\mu} + \mu \int_0^1 \frac{\Psi(\mu')}{\mu - \mu'} [Y(\mu) X(\mu') - X(\mu) Y(\mu')] d\mu',$$

where $\Psi(\mu)$ is an even polynomial in μ such that

$$\int_0^1 \Psi(\mu) d\mu \leq \tfrac{1}{2},$$

and can be solved numerically by methods similar to those used for the functional equation of paper XIV.

In the "conservative" case of perfect scattering, when $\int_0^1 \Psi(\mu) d\mu = \tfrac{1}{2}$, it turns out that the solutions are not unique; if $X(\mu), Y(\mu)$ is one solution, another is $X(\mu) + Q\mu[X(\mu) + Y(\mu)]$,

$Y(\mu) - Q\mu[X(\mu) + Y(\mu)]$ for any constant Q. This arbitrariness can be removed in the physical problem by appealing to a known formula for the integral $\int_{-1}^{1} I(\tau, \mu)\mu^3 d\mu$.

The results obtained are applied to the cases of (i) isotropic scattering with an albedo $\omega_0 \leqq 1$, (ii) Rayleigh scattering, (iii) scattering with the phase function $\lambda(1 + x\cos\theta)$, (iv) Rayleigh scattering with polarization taken into account.

             F. Smithies (Cambridge, England).

Rozovskii, M. I. On the integro-differential telegraph equations. Doklady Akad. Nauk SSSR (N.S.) **59**, 1265–1268 (1948). (Russian)

The author considers the pair of integro-differential equations

$$\frac{\partial V}{\partial x} = RJ + L\frac{\partial J}{\partial t} + \int_{-\infty}^{t} \varphi(t-\tau)\frac{\partial J(\tau)}{\partial \tau}d\tau,$$

$$\frac{\partial J}{\partial x} = GV + C\frac{\partial V}{\partial t} + \int_{-\infty}^{t} \psi(t-\tau)\frac{\partial V(\tau)}{\partial \tau}d\tau,$$

satisfied by the potential V and the current J in a long thin conductor when electromagnetic after-effects are taken into account. He obtains a complete solution for the forced oscillations induced by an applied periodic e.m.f., and then considers the transient effects arising when a constant e.m.f. is suddenly applied. In the latter case he assumes that the transient part of the potential can be expressed in the form

$$V_2(x, t) = \sum_{k=1}^{\infty} T_k(t) \sin(k\pi x/l),$$

and shows that the functions $T_k(t)$ satisfy an integro-differential equation of the form

$$a_1\frac{d^2 T_k}{dt^2} + a_2\frac{dT_k}{dt} + \left(\frac{k^2\pi^2}{l^2} + a_3\right)T_k$$
$$+ \int_0^t \left\{\psi_1(t-s)\frac{d^2 T_k(s)}{ds^2} + \psi_2(t-s)\frac{dT_k(s)}{ds}\right\}ds = 0.$$

He then states, with an outline proof, that the general solution of this can be written

$$T_k(t) = e^{-bt}[A_1 U_1(t, k^2\pi^2/l^2) + A_2 U_2(t, k^2\pi^2/l^2)],$$

where U_1 and U_2 are functions for which explicit formulae are given in terms of certain kernels and their resolvents. The paper concludes with explicit formulae for the potential and the current.   *F. Smithies* (Cambridge, England).

Functional Analysis

✳ **Hille, Einar. Functional Analysis and Semi-Groups.** American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, New York, 1948. xii+528 pp. $7.50.

This book is devoted to the study of semi-groups (associative multiplicative systems for which no cancellation rules are postulated) and their linear representations in Banach spaces. Commutative semi-groups receive generous but not exclusive attention. Much important material contained in this volume has not been published elsewhere. Certain well-known results about groups and their representations in Banach and Hilbert spaces appear here as special cases of results about semi-groups. The book is divided into three parts. The first (Functional analysis) deals with basic concepts and methods, the second (Analytical theory of semi-groups) with the development of the central theme and the third (Special semi-groups) with applications and illustrations of the general theory. Each chapter is prefaced by a brief paragraph of orientation. An appendix is devoted to a discussion of various topics in the theory of Banach algebras. There are included a bibliography, an index and a list of special symbols.

The first part gives an extensive treatment of topics which are, in the main, familiar to workers in functional analysis, but are to be found in the literature only as isolated and scattered contributions (if at all). The contents may be indicated briefly: Chapter I, Abstract spaces (topological, additive, linear and algebraic); II, Linear operations (various types of transformation in spaces of the kinds just listed: continuous, linear, real linear, bounded linear; and topological algebras of endomorphisms of a Banach space); III, Vector-valued functions (integration, complex function theory); IV, Functions on vectors to vectors (polynomial functions, differentiation, analyticity); V, Analysis in Banach algebras (regular and singular elements, functions with values in a B-algebra and their calculus, resolvents and spectra, the exponential and logarithmic functions, homomorphisms into the complex field).

Part two starts with chapter VI, Subadditive functions (functions characterized by the inequality $f(t_1 + t_2) \leqq f(t_1) + f(t_2)$, which is shown to have implications concerning boundedness, rate of growth, continuity and differentiability), and VII, Semi-Modules (Abelian semi-groups, written additively, with special emphasis on those contained in a real n-dimensional vector space E_n; connections of this special case with the theory of subadditive functions). In chapter VIII, Addition theorems in a Banach algebra, the general problem of solving the functional equation $F(x+y) = G[F(x), F(y)]$, where G is a given analytic function of two complex variables and F is to have domain contained in a given complex Banach space and range contained in a given complex Banach algebra, is discussed, with particular reference to the case, important for the theory of semi-groups, where $G(u, v) = uv$ and the equation formally has exponential solutions $Ae^{U(x)}$, $U(x)$ being linear in x and A idempotent. The results depend in an essential way upon the topology associated with the range-space for the solution sought (it should be noted that a Banach algebra has several associated topologies of interest to the analyst). In chapter IX, Semi-Groups in the strong topology, attention is directed to the case of one-parameter semi-groups of transformations in Banach space, considered in the strong topology; important results concerning measurability, continuity, differentiability, and the exponential representation e^{tA} are established. Useful tools of analysis are introduced in chapter X, Laplace integrals and binomial series. These tools are immediately employed in chapter XI, Generator and resolvent; they give meaning to the formal relations $(\lambda - A)^{-1} = \int_0^\infty e^{-\lambda\xi}e^{\xi A}d\xi$, $e^{tA} = (2\pi i)^{-1}\int_\Gamma e^{\lambda t}(\lambda - A)^{-1}d\lambda$ between the generator A and the resolvent $(\lambda - A)^{-1}$ associated with the semi-group $T(\xi) = e^{\xi A}$. In chapter XII, Generation of semi-groups, the converse problem of setting up conditions under which a given operator A generates a one-parameter semi-group $T(\xi) = e^{\xi A}$ is examined. The case of analytic semi-groups depending on one complex parameter is examined in chapter XIII, Analytical semi-groups; of particular interest here is the extension of such a semi-group by analytic continuation to an enlarged parameter-domain.

Important and very interesting connections between superficially unrelated topics are developed in chapter XIV, Semi-Groups, ergodic theory, and Tauberian theorems. The interrelations depend upon the following formal considerations: ergodicity for a semi-group $T(\xi)$ means the existence of some generalized limit for $T(\xi)$ as $\xi \rightarrow +\infty$; special interest attaches to the Cesàro limit of order α,

$$\lim_{\xi \to +\infty} \alpha \xi^{-\alpha} \int_0^{\xi} (\xi - \tau)^{\alpha-1} T(\tau) d\tau;$$

however, the Abel limit,

$$\lim_{\lambda \to +0} \lambda \int_0^{\infty} e^{-\lambda \xi} T(\xi) d\xi = \lim_{\lambda \to +0} \lambda(\lambda - A)^{-1}$$

is easier to investigate; and the passage from the Abel limit to the Cesàro requires arguments of a Tauberian character based on the formula

$$\lambda \int_0^{\infty} e^{-\lambda \xi} T(\xi) d\xi = \{\lambda^{\alpha+1}/\Gamma(\alpha)\} \int_0^{\infty} e^{-\lambda \tau} \left\{ \int_0^{\tau} (\tau-\xi)^{\alpha-1} T(\xi) d\xi \right\} d\tau.$$

The spectral theory and the operational calculus, together with related topics, are the subject of the last chapter of part two, namely chapter XV, Spectral theory.

The special instances given in the third part of the book constitute an impressive array. The chapters may be listed as follows: XVI, Translations and powers (the semi-groups $f(x) \rightarrow f(x+\xi)$, $f(x) \rightarrow g(x)^{\xi} f(x)$ in certain function-spaces); XVII, Trigonometric semi-groups (applications to factor-sequences for Fourier series); XVIII, Semi-Groups in $L_p(-\infty, \infty)$ (applications to factor-functions for Fourier transforms and to factor-sequences for Hermitian series); XIX, Semi-Groups in Hilbert space (especially semi-groups of self-adjoint operators); XX, Semi-Groups and partial differential equations (semi-groups as the mathematical expression of the principle of determinacy; partial differential equations of physical significance belonging to each of the three standard types: hyperbolic, parabolic, and elliptic); XXI, Summability, stochastic processes, fractional integration (applications to Abel summability, Hausdorff summability, Markov chains, stochastic processes, the integrals of Riemann-Liouville, and the integrals of M. Riesz in the theory of the wave equation).

The appendix, chapter XXII, Notes on Banach algebras, covers a number of topics, including the well-known Gelfand representation theory, the theory of the radical and the theory of the exponential function.

It is impossible in a review of this kind to give an adequate account of the wealth of material covered in the book, or to emphasize the interest of specific results. It is appropriate to note the systematic, incisive and polished character of the author's treatment. His scholarly handling of the subject will be appreciated by all users of the book, as will the clarity of his expository style. The general character of the work is analytical rather than algebraic, although algebraic concepts and methods are everywhere given due prominence (as they must be in any adequate discussion of semi-groups). *M. H. Stone* (Chicago, Ill.).

Aleksandrov, P. S. On the so-called quasiuniform convergence. Uspehi Matem. Nauk (N.S.) **3**, no. 1(23), 213–215 (1948). (Russian)

The author proves the following theorem. Let a convergent sequence $f_1, \cdots, f_n, \cdots$ of continuous mappings from a topological space X into a metric space Y be given.

In order that the limit f of the sequence be continuous it is necessary and sufficient that, for each positive ϵ and each natural number N, there can be found open sets $\Gamma_0, \Gamma_1, \cdots, \Gamma_k, \cdots$ and natural numbers $n_0, n_1, \cdots, n_k, \cdots$ satisfying the following conditions: (a) the sum of all the open sets Γ_k is all of X, (b) each $n_k > N$ and (c) the distance from $f(x)$ to $f_{n_k}(x)$ is less than ϵ for all x in Γ_k. *M. M. Day.*

Riss [Riesz], F. On some fundamental notions of the general theory of linear operators. Uspehi Matem. Nauk (N.S.) **1**, no. 2(12), 147–178 (1946). (Russian)

Translated from Ann. of Math. (2) **41**, 174–206 (1940); these Rev. **1**, 147.

Wintner, Aurel. On Dirac's theory of continuous spectra. Physical Rev. (2) **73**, 781–785 (1948).

The author discusses an argument used by Dirac to prove that the spectrum of every quantum-mechanical operator contains the entire real axis. He shows that this cannot be true in general, and must be replaced by the statement that the continuous spectrum of such an operator is either void or unbounded. He carries through the proof of this statement for wave equations of the form $(p\varphi')' + (\epsilon + q)\varphi = 0$ for the interval $(0, \infty)$. The differential equation is transformed into an integral equation of the form $\int_0^{\infty} K(s, t)\varphi(t) dt = \lambda \varphi(s)$, where $\lambda = 1/\epsilon$, and results of Weyl [Math. Ann. **68**, 220–269 (1910)] on the spectrum of a bounded integral operator are then used to complete the proof.

[Reviewer's note. The result, and possibly more general results, can be proved by using von Neumann's theorem [Characterisierung des Spektrums eines Integraloperators, Actualités Sci. Ind., no. 229, Hermann, Paris, 1935] that for a self-adjoint integral operator the point 0 is either a limit-point of the spectrum or a point of infinite multiplicity of the point spectrum.] *F. Smithies.*

Orlicz, W. Sur les opérations linéaires dans l'espace des fonctions bornées. Studia Math. **10**, 60–89 (1948).

The space $M_I(a, b)$ is defined as the space of real-valued measurable functions essentially bounded in an interval (a, b) in which the symbol $x_n \overset{i}{\rightarrow} x_0$ means (1) $x_n(t)$ are uniformly essentially bounded in (a, b); (2) $x_n(t) \rightarrow x_0(t)$ in measure on (a, b). An operator U is called (M_I, Y) linear if it is additive and continuous from M_I to a space Y of type (F). Representations of certain such operators have been given by Alexiewicz [Ann. Soc. Polon. Math. **19**, 140–160 (1947); these Rev. **9**, 96]. In this paper the author gives various types of sufficient conditions to insure the continuity of an additive operator U from M_I to Y. He considers in particular (F) spaces Y satisfying a certain property (Z) and spaces of real functions satisfying a property (Z_a). Here Y has property (Z) if for any sequence $\{y_i\}$ having the property that for every sequence $\{\epsilon_i\}$, $\epsilon_i = 0$ or 1, the set $\{\sum_{i=1}^{n} \epsilon_i y_i\}$ is bounded, then $\sum_{i=1}^{\infty} y_i$ exists. If Y is a space of real functions, Y has property (Z_a) if for every sequence $\{y_i(t)\}$ such that for every $\tau = \{\epsilon_i\}$, $\epsilon_i = 0$ or 1, $\lim_{n \to \infty} \sum_{i=1}^{n} \epsilon_i y_i(t) = y_n(t)$ in measure then $\sum_{i=1}^{\infty} y_i$ exists in Y. Many well-known function spaces have both properties.

For any space Y with property (Z) a sufficient condition that U be (M_I, Y) linear is $x_n \overset{i}{\rightarrow} x$ implies $\lim \inf_{n \to \infty} \|Ux_n\| \geq \|Ux_0\|$. If Y is a function space with property (Z), then if $x_n \overset{i}{\rightarrow} x_0$ implies $Ux_n \rightarrow Ux_0$ in measure, U is (M_I, Y) linear. If Y has property (Z_a) U is (M_I, Y) linear if (1) $y_n(t) \rightarrow y_0(t)$ in measure implies $\lim \inf_{n \to \infty} \|y_n\| \geq \|y_0\|$, (2) $x_n \rightarrow x_0$ implies $Ux_n \rightarrow Ux_0$ in measure. The author applies these theorems

and others of the same character to proving that certain types of integral operators from M_l to various function spaces are (M_l, Y) linear.

In the latter half of the paper he gives generalizations of the Banach-Steinhaus theorem to sequences of operators which are (M_l, Y) linear. If U_n is a sequence of operators (M_l, Y) linear and if $Ux = \lim_{n \to \infty} U_n x$ for all $x \in M_l$ then U is (M_l, Y) linear. For function spaces satisfying (Z_n) we usually need only $U_n x \to Ux$ in measure. The author concludes by giving certain sufficient conditions for continuity of certain biadditive operators.   *R. E. Fullerton.*

✳**Shilov, G. On regular normed rings.** Trav. Inst. Math. Stekloff **21**, 118 pp. (1947). (Russian. English summary)

Let R be a commutative normed ring and $\mathfrak{M}$ its space of maximal ideals topologized in the usual way [Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51] and let $x \to x(M)$ be the homomorphism of R onto the complex numbers associated with $M \in \mathfrak{M}$. The ring R is said to be "regular" provided, for closed $F \subset \mathfrak{M}$ and $M_0 \in \mathfrak{M} - F$, there exists $x \in R$ such that $x(M) = 0$ for $M \in F$ and $x(M_0) \neq 0$. The present monograph is devoted to a detailed investigation of regular normed rings. The results are applied to the study of a number of important examples giving, e.g., several results of Ditkin [Uchenye Zapiski Moskov. Gos. Univ. Matematika **30**, 83–130 (1939); these Rev. **1**, 336] including generalizations of Wiener's Tauberian theorem.

If R contains only one maximal ideal, then it is called a "primary ring." An ideal contained in only one maximal ideal is a "primary ideal." If I is a primary ideal, then the residue ring R/I is a primary ring. It is shown that every maximal ideal in a regular ring without radical contains a minimal closed primary ideal. Let R be a regular ring without radical and let $||x||_{M_0} = \inf ||y||$ for all $y \in R$ such that $y(M) = x(M)$ in some neighborhood of M_0. Set $|||x||| = \sup ||x||_M$; then R is said to be of "type C" provided the norm $|||x|||$ is equivalent to the original norm in R.

Let S be a bounded closed set of complex numbers and to each $t \in S$ let there be given a primary ring K_t with maximal ideal M_t and one generator $z_t \in M_t$ (i.e., the smallest closed subring of K_t containing z_t and the identity e_t is K_t itself). The "continuous direct sum" $\sum K_t$ is defined as the ring of all functions $x(t)$ defined on S with $x(t) \in K_t$ such that the image of $x(t)$ in K_t/M_t is a continuous function of t and $||x(\cdot)|| = \sup_t ||x(t)|| < \infty$, under the natural algebraic operations and norm $||x(\cdot)||$. Denote by $\sum' K_t$ the subring of $\sum K_t$ generated by $z(t) = t e_t + z_t$. The following structure theorem is obtained. Let R be a ring of type C with one generator ($\mathfrak{M}$ here is a bounded closed set of complex numbers); then $R = \sum' R/J_M$, where J_M is the minimal closed primary ideal contained in M. In the finite dimensional case, there are only a finite number of terms and the sum is the ordinary direct sum.

Rings of power series in the sense of the following definition are also studied in some detail. A ring K with one generator z is called a "ring of power series" provided it is isomorphic with a ring of formal power series $\sum_{n=0}^{\infty} a_n \zeta^n$ such that z corresponds to ζ and, if x, x_m correspond respectively to $\sum a_n \zeta^n$, $\sum a_n^{(m)} \zeta^n$ and if $x_m \to x$ in K, then $a_n^{(m)} \to a_n$ for each n. A ring K with one generator z is a ring of power series if, and only if, the closed ideals generated respectively by z, z^2, $\cdots$, z^n, $\cdots$ intersect in only the zero element, the representation as a power series being with respect to z.

The summary is quite complete [15 pages] and covers essentially everything except detailed proofs.

          C. E. Rickart (New Haven, Conn.).

✳**Lusternik, L. Topology of functional spaces and calculus of variations in the large.** Trav. Inst. Math. Stekloff **19**, 100 pp. (1947). (Russian. English summary)

This paper is chiefly concerned with the results obtained by applying cohomology theory, and in particular the duality principle and the Pontrjagin removing theorem, to the problems of analysis in the large. Chapter I presents the analysis in the large of functions on topological spaces. There are certain applications in chapter II to nonlinear integral equations; but the principal aim is the development of theorems concerning the existence of geodesics in various curve-families on certain manifolds, notably in the family of curves joining two fixed points of a homeomorph of a sphere and the family of closed curves on a homeomorph of an n-sphere.

The author uses a number of topological terms without formal definition, which introduces difficulty especially when these terms are currently assigned several nonequivalent meanings. In particular, "upper cycle" (i.e., cocycle) is undefined. In the applications, the author applies the name to objects which do not satisfy any definition of cocycle known to the reviewer. More particularly, these objects do not seem identical with those concerning which P. S. Alexandroff has proved [Trans. Amer. Math. Soc. **54**, 286–339 (1943); these Rev. **5**, 48] the general form of the Pontrjagin removing theorem, which plays a vital role in the present paper.   *E. J. McShane* (Charlottesville, Va.).

Calculus of Variations

Lévy, Paul. Le problème des cols en calcul des variations. Bull. Soc. Math. France **75**, 31–42 (1947).

Let $\Phi(e)$ be defined and continuous on a topological space Ω. A subset ω of Ω will be called a "col" if $\Phi(e)$ is equal to a constant c on ω and the set on which $\Phi(e) < c$ is divided into two or more parts by ω and the set on which $\Phi(e) > c$. The author discusses closed geodesics on closed surfaces and on polyhedra that define "cols," giving detailed descriptions of those that lie on the regular tetrahedron, the cube and the regular octahedron. Similar results are given for the geodesics on an ellipsoid joining two given points and for the problem of finding the surface of revolution of minimum area.   *M. R. Hestenes* (Los Angeles, Calif.).

Shiffman, Max. A theory of minimax. Proc. Nat. Acad. Sci. U. S. A. **34**, 96–101 (1948).

The author shows that if $y = \varphi(x)$ is an extremal for a regular integral

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx$$

and n is the number of nonpositive characteristic numbers, then there exist n integrals $J_i(y)$ and an n-parametric family of arcs $y = \varphi(x, c_1, \cdots, c_n)$ containing $y = \varphi(x)$ for $c_i = 0$ such that $I(y) \geq I(\varphi(x, c))$ for all neighboring arcs joining the end point of $y = \varphi$ and having $J_i(y) = c_i$. Moreover, a formula for $I(\varphi(x, c))$ is given. The author discusses the question of reducibility. These results are of importance in the study of index theorems in the calculus of variations in the large and in the small.   *M. R. Hestenes.*

Baiada, Emilio. Sopra un problema non regolare e un problema isoperimetrico del calcolo delle variazioni. Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 59–75 (1948).

In a sequence of five papers [Trans. Amer. Math. Soc. **44**, 429–438, 439–453 (1938); **45**, 151–171, 173–196, 197–216 (1939)] the reviewer established existence theorems for certain non-regular problems and isoperimetric problems in the plane. The author shows that the theorem of note III, on nonregular problems, can be established by methods similar to those of Tonelli, with gain in simplicity and also with weakened hypotheses concerning the boundary of the region containing the admissible curves. To show that his theorems on isoperimetric problems were not included in earlier results, the reviewer devised an example in note V. The author shows that this example can be treated by an extension (not trivial) of methods used by Tonelli.

E. J. McShane (Charlottesville, Va.).

Cinquini, Silvio. Sopra i problemi variazionali in forma parametrica dipendenti dalle derivate di ordine superiore. Ann. Scuola Norm. Super. Pisa (2) **13** (1944), 19–49 (1948).

The author derives some theorems on the existence of a minimum for curvilinear integrals in the plane whose integrands depend on the curvature as well as the position and direction of the curve. The proofs are based on the standard method of lower semi-continuity. Existence theorems are given first for bounded domains and then for unbounded domains. *L. M. Graves* (Chicago, Ill.).

De Donder, Th., et van den Dungen, F. H. La formule fondamentale du calcul des variations écrite en variables canoniques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 9–16 (1948).

In the present paper the Euler equations of an integral in the calculus of variations are derived in canonical form in terms of a Hamiltonian function. The authors consider the case in which the integrand is a function of m independent variables, n dependent variables and derivatives of an arbitrary order. *M. R. Hestenes.*

Kimball, W. S. Stokes' theorem and the vector integrand for line integrals in the calculus of variations. Philos. Mag. (7) **38**, 842–879 (1947).

This paper contains some formal manipulations connected with the calculus of variations. The significance of the author's discussion is not clear to the reviewer.

L. M. Graves (Chicago, Ill.).

Karush, William. A semi-strong minimum for a multiple integral problem in the calculus of variations. Trans. Amer. Math. Soc. **63**, 439–451 (1948).

Consider an integral $I(z) = \int_A f(x, z, p)dx$, where x is m-dimensional, z is one-dimensional and $p_i = \partial z/\partial x_i$. Suppose that $z(x)$ (x on A) is a function of class C' that satisfies the Euler equation in integral form and a semi-strong condition of Weierstrass, and is such that $I_2(\eta) \geq k \int_A \eta^2 dx$, where $k > 0$ and $I_2(\eta)$ is the second variation of $I(z)$. The author shows that under these conditions $z(x)$ affords a semistrong relative minimum to $I(z)$ on the class of functions having the same boundary values as $z(x)$. This paper is one of few concerned with sufficiency theorems for multiple integrals and is the first that uses an indirect method. The ideas presented in this paper have been extended more recently

by Hestenes [see the following review] and by H. Meyer [dissertation, University of Chicago], so as to obtain sufficiency theorems for a strong relative minimum.

M. R. Hestenes (Los Angeles, Calif.).

Hestenes, Magnus R. Sufficient conditions for multiple integral problems in the calculus of variations. Amer. J. Math. **70**, 239–276 (1948).

The author gives a set of sufficient conditions for a strong relative minimum for an isoperimetric multiple integral problem in the calculus of variations with m independent variables and one dependent variable. His proof makes no use of field theory or expansion methods but instead is based upon an indirect type of argument which has been used recently by him for the problem of Bolza.

J. E. Wilkins, Jr. (Buffalo, N. Y.).

Van Hove, Léon. Sur la construction des champs de De Donder-Weyl par la méthode des caractéristiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 278–285 (1946).

For the problem of minimizing a multiple integral in the calculus of variables with m independent and n dependent variables, the notion of a geodesic field was introduced by De Donder [C. R. Acad. Sci. Paris **156**, 609–611, 868–870 (1913)] and discussed by Weyl [Ann. of Math. (2) **36**, 607–629 (1935)]. The author solves the partial differential equations defining such a geodesic field by the method of characteristics. *J. E. Wilkins, Jr.* (Buffalo, N. Y.).

Van Hove, Léon. Sur les champs de Carathéodory et leur construction par la méthode des caractéristiques. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **31** (1945), 625–638 (1946).

For the problem of minimizing a multiple integral in the calculus of variations with m independent and n dependent variables, Carathéodory [Acta Univ. Szeged. Sect. Sci. Math. **4**, 193–216 (1929)] has introduced a field which is distinct (when $m > 1$) from that of De Donder and Weyl; cf. the preceding review]. The author shows how to reduce the construction of a Carathéodory field to that of a De Donder-Weyl field by means of a change of dependent and independent variables. The construction of the De Donder-Weyl field may then be carried out by the method of characteristics [cf. the preceding review]. *J. E. Wilkins, Jr.*

Stampacchia, Guido. Alcuni teoremi sull'estremo assoluto degli integrali doppi del calcolo delle variazioni dipendenti dalle derivate del secondo ordine. Giorn. Mat. Battaglini (4) **1**(77), 36–54 (1947).

Let $I[u]$ be the integral $\iint f(x, y, u, u_{xx}, u_{yy})dxdy$ over a region D bounded by finitely many sectionally smooth closed curves. Functions admitted are those continuous with their first partials, having u_x absolutely continuous on almost all abscissas and u_y on almost all ordinates, with summable u_{xx} and u_{yy}. These constitute "class W"; if $u \in W$ and on the boundary curves $x = x(s)$, $y = y(s)$ of D the functions x', y' and the tangential and normal derivatives of u all satisfy a Hölder condition, then u is of "class W^*." An equicontinuity theorem is deduced for families of functions u whose Laplacians $\Delta_2 u$ satisfy $\iint |\Delta_2 u|^{2+\alpha} < \text{constant}$. The classes W and W^* are metrized by the distance function $\max [\sup |u - \bar{u}|, \sup |u_x - \bar{u}_x|, \sup |u_y - \bar{u}_y|]$. With this metric, $I[u]$ is lower semicontinuous if $f(x, y, u, w)$ is convex and nonlinear in w for each (x, y, u). If $f(x, y, u, w) \geq u|w|^{2+\alpha} + N$ with positive u and α, and f is convex in w for each (x, y, u), the minimum of $I[u]$ is attained on each

closed uniformly bounded subclass of W^*. An analogue holds for subclasses of W; and an extension to integrals $\iiint f(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy})dxdy$ is indicated.

E. J. McShane (Charlottesville, Va.).

Reid, William T. Comments on a paper of J. Abdelhay. Anais Acad. Brasil. Ci. 20, 57–61 (1948).

The paper appeared in the same Anais 17, 45–49 (1945); these Rev. 7, 67. The author amplifies comments which he made in the cited review.

Theory of Probability

✱**Wiener, Norbert.** Cybernetics, or Control and Communication in the Animal and the Machine. Actualités Sci. Ind., no. 1053. Hermann et Cie., Paris; The Technology Press, Cambridge, Mass.; John Wiley & Sons, Inc., New York, 1948. 194 pp. $3.00.

The author has undertaken the ambitious program of making a survey of the field of control and communication (the word cybernetics is taken from a Greek word meaning steersman) from the point of view of modern man and society. The following list of chapter headings will give an idea of the scope of the book. (I) Newtonian and Bergsonian time; (II) Groups and statistical mechanics; (III) Time series, information, and communication; (IV) Feed-Back and oscillation; (V) Computing machines and the nervous system; (VI) Gestalt and universals; (VII) Cybernetics and psychopathology; (VIII) Information, language, and society. Mathematical Reviews is not the place to evaluate the author's success in tying together these disparate subjects. However, it should be noted that feedback, by means of which a mind (individual or group) or a machine is enabled to modify its activities in the light of its previous activities, plays a central role. Of specifically mathematical interest is the discussion in chapters II–IV. Chapter II is a discussion of statistical mechanics (ergodic theory mathematically), with illuminating comments on the logical place and necessity of such concepts as transformation group and invariant. Chapter III is devoted to stationary stochastic processes. In this chapter, if x is a random variable with probability density $f(x)$, $\int f(x) \log_2 f(x) \, dx$ is "the amount of information associated with the density," that is, with the x-distribution. This quantity (which is the negative of entropy in another context) has suggestive properties which justify its name. For example, the random variable $\varphi(x)$ has at most the same amount of information as x. Particular attention is paid to the stationary processes which can be expressed in terms of the Brownian motion process, and to prediction theory. If $x(t)$ is a sample function of a stationary process, the evaluation of the best linear least squares prediction of $x(t+h)$ in terms of $x(s)$ for $s \leq t$ is sketched. [Cf. Wiener, Bull. Soc. Mat. Mexicana 2, 37–42 (1945); these Rev. 7, 461; cf. also Kolmogoroff, Bull. Acad. Sci. URSS Sér. Math. [Izvestia Akad. Nauk SSSR], 5, 3–14 (1941); these Rev. 3, 4, for similar material from a different point of view.] If one predicts $x(t+h)$ from the past of $x(t)+y(t)$, where the $\{x(t), y(t)\}$ process is a two-dimensional stationary process, prediction theory becomes filter theory, and the amount of information integral is applied to give a plausible measure of the information which can be filtered from a mixture of message and noise. Chapter IV gives a suggestive introduction to the mathematical and practical

theory of servo-mechanisms, including a discussion of the problem of stabilization by means of one or more feed-backs.

The present book is one of a projected series; the others are to cover various specific applications of cybernetics, and it is only with their appearance that the full power of the method (other than as a helpful analogical tool) will become clear. The author anticipates that cybernetics, essential as it is in studies ranging from the design of machines to the analysis of the learning process of the brain, will profoundly change modern life. He is pessimistic, however, on the future of human society and in particular on the possibility of the creation of a precise science of sociology or economics. *J. L. Doob* (Urbana, Ill.).

Doob, J. L. Asymptotic properties of Markoff transition probabilities. Trans. Amer. Math. Soc. 63, 393–421 (1948).

Doeblin's or Kryloff and Bogoliuboff's conditions under which the well-known decomposition theorem for Markov chains is valid are not fulfilled in the case of some relatively simple processes such as the renewal process. The author shows that under a weaker condition (but assuming the existence of a self-reproducing distribution) another decomposition theorem is applicable: the general decomposition of stationary processes into metrically transitive processes is adapted to the stationary Markov case; a direct proof is given and stronger results are obtained in this case. They are strengthened when the singular component of the transition probability with respect to the self-reproducing measure vanishes asymptotically; conditions are given under which the $(C, 1)$ limits can be replaced by ordinary limits; Blackwell's decomposition theorem is applied. The restriction on the singular component when valid for all x and Doeblin's condition are compared. *M. Loève.*

Doob, J. L. Renewal theory from the point of view of the theory of probability. Trans. Amer. Math. Soc. 63, 422–438 (1948).

Let $x(t)$ be the age of the population at time t and $U_x(t)$ be the expected number of deaths before time t when the initial age is x, under the usual assumptions of the renewal theory. Feller showed that $\lim_{t\to\infty} U_0(t)/t$ exists and Täcklind gave asymptotic expressions for $U_0(t)$ under assumptions on the moments of the lifetime distribution. The author uses the strong law of large numbers for equidistributed independent random variables to prove Feller's results and also that $x(t)/t \to 1$ with probability one. Results of the paper reviewed above are applied to the renewal process $x(t)$, shown to be a Markov process, stationary for a suitable initial distribution. Conditions are given under which the renewal process becomes asymptotically the corresponding stationary process. *M. Loève* (Berkeley, Calif.).

Arnous, Edmond, et Massignon, Daniel. Les principales familles de fonctions aléatoires et leurs propriétés. C. R. Acad. Sci. Paris 226, 785–787 (1948).

In previous notes [same vol., 318–320, 557–559 (1948); these Rev. 9, 398, 399] the authors had given necessary and sufficient conditions that a quantum mechanical system (defined by its time evolution and other operators) should correspond to a stochastic process of the second order. The present note gives theorems (in terms of the metric of convergence in the mean) covering the continuity, differentiability, and ergodicity of the corresponding stochastic processes. *B. O. Koopman* (New York, N. Y.).

Arnous, Edmond, et Massignon, Daniel. Équations d'évolution des lois de probabilité et théorie du transfert pour les fonctions aléatoires du second ordre. C. R. Acad. Sci. Paris 226, 1127–1129 (1948).

The authors give, in the language of their previous papers [cf. the preceding review], the equation of transfer and other known equations [involving a random variable $x(t)$ depending on time t, its time derivative $x'(t)$ and the various related conditional probability functions; cf. Bass, Revue Sci. 83, 3–20 (1945); these Rev. 7, 460]. *J. L. Doob.*

Laplume, Jacques. Sur le nombre de signaux discernables en présence du bruit erratique dans un système de transmission à bande passante limitée. C. R. Acad. Sci. Paris 226, 1348–1349 (1948).

The author finds an expression for the definition of a message, number of measurements per unit time, by equating the maximum possible number of identifiably different messages with the maximum possible number of identifiably different spectra. It is not made clear why these two quantities should be equal. *J. L. Doob* (Urbana, Ill.).

Mathematical Statistics

Malécot, G. Les critères statistiques et la subjectivité de la connaissance scientifique. Ann. Univ. Lyon. Sect. A. (3) 10, 43–74 (1947).

A methodological and philosophical discussion of the principles underlying statistical inference, estimation, etc. The author starts with Bayes's rule, discusses the method of maximum likelihood, the ideas of Jeffreys and de Finetti, and finally analyzes the Neyman approach. *W. Feller* (Ithaca, N. Y.).

Perks, Wilfred. Some observations on inverse probability including a new indifference rule. J. Inst. Actuar. 73, 285–312; discussion, 313–334 (1947).

The foundations of inference from the points of view of philosophy and statistical estimation. Proposal of a principle of a priori probability to replace the classical equidistribution, which lacks invariance under changes of variable. The new principle is equivalent to equidistribution in terms of a variable for which the information (in Fisher's sense) is constant. Thus this proposal is a special case of that of Jeffreys [Proc. Roy. Soc. London. Ser. A. 186, 453–461 (1946); these Rev. 8, 163]. Discussion of alternatives and of application to actuarial problems. *J. W. Tukey.*

Dumas, M. Sur les courbes de fréquence de K. Pearson. Biometrika 35, 113–117 (1948).

Pearson discussed some types of curves satisfying the differential equation $y'/y = (x-a)/(c_0 + c_1 x + c_2 x^2)$. He divided the plane of values of the statistical moment parameters β_1, β_2 into several regions and identified with each region one of his now famous types of curves. The author demonstrates that within one of these types may occur several qualitatively distinct curves. No less than some thirty-four figures would be required to adequately illustrate the available variation. Particularly noteworthy are the special cases marking common boundary types. In the study this author introduces "the biquadratic of discontinuity of derivatives," which is seen to play an essential role. *A. A. Bennett.*

Guttman, Louis. An inequality for kurtosis. Ann. Math. Statistics 19, 277–278 (1948).

If $Et = 0$, $Et^2 = 1$, and $Et^4 = 1 + \alpha^2$, then

$$\Pr\{1 - \theta \leq t^2 \leq 1 + \theta\} > 1 - \theta^2 \alpha^{-2}$$

for any $\theta > 0$. Also

$$\Pr\{1 - \theta \leq t^2 \leq 1 + \theta + 2\alpha^2 \theta^{-1}\} \geq 1 - (1 + \theta^2 \alpha^{-2})^{-1}$$

and $\Pr\{1 - \theta - 2\alpha^2 \theta^{-1} \leq t^2 \leq 1 + \theta\} > 1 - (1 + \theta^2 \alpha^{-2})^{-1}$ for any $\theta > 0$. In particular, $\Pr\{|t| < (1+\theta)^{\frac12}\} \geq 1 - (1 + \theta^2 \alpha^{-2})^{-1}$. These follow from Chebyshev's inequality applied to $t^2 + k$. *J. W. Tukey* (Princeton, N. J.).

Rothschild, Colette, et Mourier, Édith. Sur les lois de probabilité à régression linéaire et écart type lié constant. C. R. Acad. Sci. Paris 225, 1117–1119 (1947).

For the linearity of regression of a random variable Y on another X and for the homoscedasticity of the distribution of Y, given $X = x$, it is necessary and sufficient that $\varphi(u, v)$, the characteristic function of X and Y, satisfy the equations $\varphi_v(u, 0) = a \varphi_u(u, 0)$ and $\varphi_{vv}(u, 0) = a^2 \varphi_{uu} - \sigma^2 \varphi(u, 0)$, where a and $\sigma > 0$ are arbitrary constants. This general result is applied to the particular case where $X = e_1 + e_2$, $Y = e_1 + e_3$, with e_1, e_2, e_3 representing independent random variables. [Conversations with one of the authors revealed that, since the note was published, they have become aware of some lack of rigor.] *J. Neyman* (Berkeley, Calif.).

Vodička, Václav. Fonctions symétriques et leur aplication dans la statistique mathématique. Acta Fac. Nat. Univ. Carol., Prague no. 174 (1947), 17–19 (1947). (Czech and French)

Summary of a thesis.

Feller, W. On the Kolmogorov-Smirnov limit theorems for empirical distributions. Ann. Math. Statistics 19, 177–189 (1948).

Let $S_N(x)$ be the empirical distribution of a sample of size N, $F(x)$ the corresponding theoretical distribution and $D_N = \sup |S_N(x) - F(x)|$. Kolmogorov proved in the case of a continuous $F(x)$ that, when $N \to \infty$, the limiting distribution of $N^{\frac12} D_N$ is given by $L(z) = [1 - 2\sum_{r=1}^{\infty} (-1)^{r-1} e^{-2r^2 z^2}]$ for $z > 0$ and, naturally, $L(z) = 0$ for $z \leq 0$. Smirnov showed that the same limiting distribution holds for $N^{\frac12} D_{m,n}$, where $D_{m,n} = \sup |S_m(x) - T_n(x)|$, $S_m(x)$ and $T_n(x)$ being the empirical distributions of two independent samples corresponding to the same continuous $F(x)$, $N = mn/(m+n)$ and m and $n \to \infty$ so that $m/n \to a$, constant.

Reducing the problem (of importance for the sampling theory) to its ultimate elements, the author gives a derivation of these propositions using generating functions and their limiting form, the Laplace transforms. Moreover he shows how, by the same method, parallel theorems may be derived for the extrema of the differences themselves and, as an example, proves that the limiting distribution of $N^{\frac12} D_N^+$ is $1 - e^{-2z^2}$, where $D_N^+ = \sup \{S_N(x) - F(x)\}$. *M. Loève* (Berkeley, Calif.).

Smirnov, N. Table for estimating the goodness of fit of empirical distributions. Ann. Math. Statistics 19, 279–281 (1948).

The table [reproduced from Bull. Math. Univ. Moscow 2, no. 2 (1939); these Rev. 1, 345] gives the values of $L(z)$ [see the preceding review] varying from .000001 ⋯ to .99999997 ⋯ for z varying from .28 to 3.00. *M. Loève.*

David, F. N. A χ^2 'smooth' test for goodness of fit. Biometrika 34, 299–310 (1947).

The author suggests, as an adjoint to the customary χ^2 test for goodness of fit, that the signs of the deviations from expected numbers be examined to see whether they change sufficiently often. The distribution of the number of alternations of sign is asymptotically the same as the distribution given by Stevens [Ann. Eugenics 9, 82–93, 315–320 (1939); these Rev. 1, 245] for the number of alternations of an event and its complement in a series of independent trials. The suggested criterion is asymptotically independent of χ^2, which permits construction of a single statistic embodying both criteria. The distribution theory has been checked for fairly small numbers by extensive empirical sampling.

G. W. Brown (Ames, Iowa).

David, F. N. A power function for tests of randomness in a sequence of alternatives. Biometrika 34, 335–339 (1947).

The nonparametric test for randomness based on the number of alternations of an event and its complement in a series of trials [W. L. Stevens, Ann. Eugenics 9, 82–93, 315–320 (1939); these Rev. 1, 245] is conditioned on the number r_1 of occurrences and the number r_2 of nonoccurrences of the event. In this paper the author investigates the behavior of the one-tailed test under the alternative hypothesis that the underlying mechanism corresponds to a simple Markov chain and presents graphically the conditional power functions for selected alternatives of this class, with $(r_1, r_2) = (14, 6)$, $(10, 10)$, $(7, 3)$ and $(5, 5)$. The actual power function, as conventionally defined, is the expected value of the conditional power function over the distribution of r_1, and is not calculated in this article.

G. W. Brown (Ames, Iowa).

Finney, D. J., and Stevens, W. L. A table for the calculation of working probits and weights in probit analysis. Biometrika 35, 191–201 (1948).

Let $P = \int (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}u^2} du$, where the integral is from $-\infty$ to $Y-5$. Let Z be the integrand at the upper limit. Let $Q = 1 - P$, which is the integral from $Y-5$ to $+\infty$, and $w = Z^2/PQ$. The following are tabulated to 4 decimals: $Y + Q/Z$ for $Y = 3.58(0.01)9.00$, $Y - P/Z$ for $Y = 1.00(0.01)6.42$, w and $1/Z$ for $Y = 1.00(0.01)9.00$. These values are useful in fitting cumulative normal curves to dosage-response data.

J. W. Tukey (Princeton, N. J.).

Neyman, J., and Scott, Elizabeth L. Consistent estimates based on partially consistent observations. Econometrica 16, 1–32 (1948).

A sequence $\{X_i\}$ $(i=1, 2, \cdots)$ of mutually independent chance variables is considered and the case is studied where the joint distribution of the X's involves infinitely many unknown parameters which can be split into two parts. The first part is composed of a finite number of parameters, say $\theta_1, \cdots, \theta_s$, each of which appears in the distribution functions of infinitely many X's, while the second part contains infinitely many parameters, say $\xi_1, \xi_2, \cdots$, each of which appears only in the distributions of a finite number of X's. If the above situation arises, the sequence $\{X_i\}$ is called partially consistent. The parameters $\theta_1, \cdots, \theta_s$ are referred to as structural parameters, while the parameters $\xi_1, \xi_2, \cdots$ are called incidental. Partially consistent observations arise and seem to be of importance in various fields of application, such as physics, astronomy, economics, etc.

The authors show by examples that maximum likelihood estimates of structural parameters obtained from a partially consistent series of observations need not be consistent; and if they are consistent, they need not possess the property of asymptotic efficiency. These are rather unexpected results. The possibilities for obtaining consistent estimates for the structural parameters are studied. While the authors do not offer a completely general solution, they describe some valuable specific methods, based on a modification of the maximum likelihood equations, which work well in many important cases. Also the asymptotic variance of such estimates is investigated.   A. Wald (New York, N. Y.).

Aspin, Alice A. An examination and further development of a formula arising in the problem of comparing two mean values. Biometrika 35, 88–96 (1948).

Let $h(s^2) = h(s_1^2, \cdots, s_k^2, P)$ be that function of (s_i^2) and P, such that the probability is P that $y - \eta$ falls short of $h(s^2)$. Here η is any population parameter estimated by an observed quantity y which in turn is normally distributed with variance $\sigma_y^2 = \sum_{i=1}^{k} \lambda_i \sigma_i^2$, where, furthermore, the data provide estimates s_i^2 of the unknown variances σ_i^2, distributed as proposed by B. L. Welch [Biometrika 34, 28–35 (1947); these Rev. 8, 394]. One may expand $h(w)$ in series of terms in $1/f_i$ (when f_i indicates degrees of freedom corresponding to σ_i), and the sum of terms of order r in $1/f_i$ may be called h_r. Then if ξ is the normal deviate, h_r/ξ is a polynomial of order r in ξ^2. In particular, Welch has secured rather formidable expressions for h_0, h_1, h_2. This article introduces appropriate compact notation and succeeds in continuing the computation through h_3 and h_4. The results of the voluminous and heavy algebraic manipulation are checked in several ways and some numerical tables are given. The basic formula originated in the study of the problem of comparing two mean values.   A. A. Bennett.

Rasch, G. A functional equation for Wishart's distribution. Ann. Math. Statistics 19, 262–266 (1948).

Author's summary: A new derivation is given of Wishart's distribution, in which a fundamental property of the multivariate normal distribution is utilized, viz. the invariance of the distribution type against a linear transformation. The same principle is used for evaluation of the constant and the determination of the moment matrix in the multidimensional normal distribution.   L. A. Aroian.

Fog, David. The geometrical method in the theory of sampling. Biometrika 35, 46–54 (1948).

The author shows how the geometrical method of obtaining sampling distributions in normal multivariate analysis may be translated into analytical form while preserving the essential simplicity of the geometrical methods. Three examples, of which the third is a derivation of Wishart's distribution, illustrate the theory.   L. A. Aroian.

Wishart, John. Proofs of the distribution law of the second order moment statistics. Biometrika 35, 55–57 (1948).

The various proofs of Wishart's distribution in multivariate analysis are discussed and compared. Some comments are made on Fog's paper reviewed above. To the references add the proofs of Sverdrup [Skand. Aktuarietidskr. 30, 151–166 (1947); these Rev. 9, 453], and Rasch [see the second preceding review].   L. A. Aroian.

Aroian, Leo A. **Note on the cumulants of Fisher's z-distribution.** Biometrika **34**, 359–360 (1947).

Points out the author's publication of cumulants of the z-distribution [Ann. Math. Statistics **12**, 429–448 (1941); these Rev. **3**, 175] prior to Wishart [Biometrika **34**, 170–178 (1947); these Rev. **8**, 474]. Discussion of type A approximation to z and type III to F.   *J. W. Tukey.*

Robbins, Herbert. **The distribution of a definite quadratic form.** Ann. Math. Statistics **19**, 266–270 (1948).

Let $U_n = \frac{1}{2}\sum_{i=1}^{n} a_i X_i^2$; the X_i are independent normal variates with zero means and unit variances and the a_i are positive constants. Let $F_n(x) = \Pr[U_n \leq x]$, $f_n(x) = F_n'(x)$, $a = (a_1 \cdots a_n)^{1/n}$, $q_i = \prod_{j \neq i}(a_j - a_i)^{-1}$ if the a_i are all distinct. Then $f_n(x) = a^{-\frac{1}{2}n} x^{\frac{1}{2}n-1} \sum_{k=0}^{\infty} c_k (-x)^k / \Gamma(\frac{1}{2}n+k)$, $x > 0$, where $c_k = \pi^{-\frac{1}{2}n} \sum \prod_{r=1}^{n} \{\Gamma(i_r + \frac{1}{2})/[i_r! a_r^{i_r}]\}$, summation over $i_1 + \cdots + i_n = k$. The c_k are computed more readily from the recurrence relations $c_0 = 1$, $\sum_{i=0}^{k} c_i c_{k-i} = a^n \sum_{i=1}^{n} q_i a_i^{-(k+1)}$ if the a_i are all distinct, with appropriate modifications in the contrary case. Another expansion expresses $f_n(x)$ as a series of χ^2 frequency functions. Corresponding expansions for $F_n(x)$ are given. The conditional distribution of U_n with $\sum X_i^2$ held constant has been discussed by von Neumann [same Ann. **12**, 367–395 (1941); these Rev. **4**, 21] and Koopmans [same Ann. **13**, 14–33 (1942); these Rev. **4**, 22].   *T. E. Harris* (Santa Monica, Calif.).

Kac, M. **On the characteristic functions of the distributions of estimates of various deviations in samples from a normal population.** Ann. Math. Statistics **19**, 257–261 (1948).

Consider $Y_n(\alpha) = n^{-1} \sum_{k=1}^{n} |X_k - \bar{X}|^\alpha$, $\alpha \geq 1$, for samples from a normal population $(0, 1)$. The author proves that the characteristic function of $Y_n(\alpha)$ is given by

$$n^{-1}(2\pi)^{-\frac{1}{2}(n+1)} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} e^{-\frac{1}{2}n x^2} e^{i/n(\frac{1}{2}|x|^\alpha + \tau n)} dx \right\}^n d\eta.$$

He also finds the explicit expression of the density function of $Y_n(1)$. The method of derivation is of interest in itself.   *M. Loève* (Berkeley, Calif.).

Jones, Howard L. **Exact lower moments of order statistics in small samples from a normal distribution.** Ann. Math. Statistics **19**, 270–273 (1948).

The means of order statistics of samples of size not exceeding 3 from a normal distribution and the second moments, pure and mixed, for samples of size at most 4 are obtained exactly by integration and the use of general identities. The results are rational in $\pi^{\frac{1}{2}}$ for the means and in terms of $3^{\frac{1}{2}}$ and π for the second moments. The variances and covariances are given for samples of size not exceeding 3. These exact results supersede some of the results of Hastings, Mosteller, Winsor and the reviewer [same Ann. **18**, 413–426 (1947); these Rev. **9**, 195]. Related results were found by Hojo [Biometrika **23**, 315–360 (1931)], who gives expansions and 5, 6 and 7 place values for the first four moments for the median and quartiles from the normal distribution in samples ranging up toward 12. Hojo also gives 8 place values of numerous related integrals. For the unit normal distribution, it is shown that $\sum_j E\{x_i x_j\} = 1$.   *J. W. Tukey* (Princeton, N. J.).

Wilks, S. S. **Order statistics.** Bull. Amer. Math. Soc. **54**, 6–50 (1948).

Let a sample of size n be drawn from a population and let the sample values be arranged in increasing order of

magnitude: $x_1 \leq \cdots \leq x_n$. Then x_r is called the rth order statistic of the sample. Corresponding definitions hold for samples from multidimensional populations. This address contains a useful expository account of the sampling theory of order statistics and their applications to various problems of statistical estimation and testing of statistical hypotheses, in the case when the population distribution function is continuous.   *H. Cramér* (Stockholm).

Moran, P. A. P. **Rank correlation and product-moment correlation.** Biometrika **35**, 203–206 (1948).

The author derives the mean and variance of Spearman's coefficient ρ_s of rank correlation in samples of n from a bivariate normal distribution with correlation coefficient ρ. It is shown that ρ_s is a biased and inconsistent estimator of ρ. A table is given comparing the expected value of ρ_s and the expected value of r, the product moment sample correlation coefficient, for various values of ρ and n. There is included a formula for the evaluation of the total probability in the positive part of the quadrivariate normal distribution.   *L. A. Aroian* (New York, N. Y.).

Nair, K. R. **The Studentized form of the extreme mean square test in the analysis of variance.** Biometrika **35**, 16–31 (1948).

If w is nonnegative, and is distributed independently of s, and if ns^2 has the chi-square distribution on n degrees of freedom, then Hartley [Biometrika **33**, 173–180 (1944); these Rev. **6**, 10] showed that the distribution function $P_n(Q)$ of w/s could be expanded in powers of n^{-1} with coefficients depending on the derivatives of $P(Q)$. The author extends the coefficients through the fourth, which involves an eighth derivative. Integrating the coefficients by parts, it is possible to express the moments of the approximation to the distribution of w/s in terms of those of w, if the derivatives of $P(Q)$ vanish fast enough at infinity. The author's formulas are, strictly speaking, meaningless, since both sides are infinite, as a result of omitting an integration by parts. The corrected formulas show that each moment is multiplied by a function of n, without regard to the distribution of w. For the first two moments these factors are $1 + (3/4n) + (25/32n^2) + (105/128n^3) + (1659/2048n^4) + \cdots$ and $1 + n^{-1} + 2n^{-2} + 4n^{-3} + 8n^{-4} + \cdots$. [Note the signs in relation to the author's formulas.]

The author considers an expansion given by Hartley for the incomplete beta function, which was based on this expansion where w^2 has the chi-square distribution, and shows that the higher terms improve the approximation. In a special case w/s reduces to Student's t, when the expansion becomes that given by Fisher [Metron **5**, no. 3, 109–111 (1925)].

Next the author considers the case where w^2 is the largest of a sample of k from the chi-square distribution with m degrees of freedom. This is the case of the largest of k variance ratios with the same denominator, which has n degrees of freedom. Finney [Ann. Eugenics **11**, 136–140 (1941); these Rev. **3**, 172] has studied this case for m even, and finds the assumption of independence of these ratios quite satisfactory for $m = 2$. The author tabulates upper 5% and 1% points of w^2/s^2 for $m = 1$ and $k = 1(1)10$ and $n = 10, 12, 15, 20, 30, 60, \infty$, to two decimals, pointing out that for large k and small n there may be substantial errors.

Passing now to the case where w^2 is the smallest of the samples of k from the chi-square on m degrees of freedom, the author points out that for $m = 2$ the independence

assumption gives a good approximation, but that it is not exact. For $m=1$ he tabulates a_0 and a_1, where the probability that $w^2/s^2 \leqq q^2$ is approximately $1 - a_0 - a_1 n^{-1}(1 - (8n)^{-1})$, for $k = 2(1)10$ and $q = 0.00(0.01)0.10$. The independence assumption proves to be very good for $q = 0.01$ which is the region of practical importance.

[Notation: k, n in review; n, ν in paper.]

J. W. Tukey (Princeton, N. J.).

Nair, K. R. The distribution of the extreme deviate from the sample mean and its Studentized form. Biometrika 35, 118–144 (1948).

Let $x_1, \cdots, x_k$ be the ordered values in a sample of k from the unit normal distribution, and let $\bar{x}$ be their mean. Let $w = x_k - \bar{x}$ and let $P_k(u)$ be the probability that $w \leqq u$. Then $P_k(u) = k^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}(k-1)} G_{k-1}(ku)$, where the G-functions defined by Godwin [Biometrika 33, 254–256 (1945); these Rev. 8, 42] are defined by $G_0(x) = 1$,

$$G_r(x) = \int_0^x G_{r-1}(t) \exp\{-t^2/2r(r+1)\} dt.$$

McKay [Biometrika 27, 466–471 (1935)] found a different expression for $P_k(u)$, and proposed the approximation $1 - P_k(u) \sim k\{1 - N((n/(n-1))^{\frac{1}{2}} u)\}$, for large u, where $N(x)$ is the cumulative for the unit normal. The upper percentage points for $k = 3(1)9$ found by this approximation deviate from the true values by at most 24×10^{-4} at 5% and 2×10^{-4} at 1%. McKay's further approximation for small u is inferior to

$$P_k(u) \sim \frac{k^{\frac{1}{2}}}{(k-1)!} \left(\frac{ku}{(2\pi)^{\frac{1}{2}}} \right)^{k-1} \left\{ 1 - \frac{k(k-1)u^2}{2(k+1)} \right\}, \quad k \geqq 2,$$

which gives very good results. Tables of $P_k(u)$ for $k = 3(1)9$, $u = 0.00(0.01)4.70$, are given to six decimals, and lower and upper 10%, 5%, 2.5%, 1%, 0.5% and 0.1% points are given for $k = 3(1)9$. These can be extended by the use of the approximations.

The author then applies Hartley's method [cf. the preceding review] to find an expansion for the distribution of w/s, where s^2 is a chi-square on n degrees of freedom, in the form $a_0 + a_1 n^{-1} + a_2 n^{-2}$. Values of a_0, a_1 and a_2 are tabulated for $k = 3(1)7$, and $w/s = 0.0(0.2)4.6$ to varying accuracy. The lower 5% and 1% points are tabulated to two decimals for $k = 3(1)9$ and $n = 10, 15, 30$ and ∞. The upper 5% and 1% points are tabulated to two decimals for $k = 3(1)9$ and $n = 10(1)20, 24, 30, 40, 60, 120$ and ∞.

Returning to $x_1, \cdots, x_k$, consider the difference between the mean of the i smallest and the mean of the j largest. The distributions of these differences can be expressed in terms of definite integrals of G-functions. For $i + j = n$ or $n - 1$ the integral is single, and for $i + j = n - 2$ it is double. In special cases these can be reduced to the distribution of the mean deviation from the median, which is stated to have been found by Godwin [unpublished].

[Notation: n, k, i, j in review; ν, n, k, l in paper.]

J. W. Tukey (Princeton, N. J.).

Radhakrishna Rao, C. Tests of significance in multivariate analysis. Biometrika 35, 58–79 (1948).

The author shows how a number of problems of multivariate analysis can be solved by the use of the method of discriminant functions. The results are essentially applications of the principles of an earlier paper [Sankhyā 7, 407–414 (1946); these Rev. 8, 162]. An asymptotic distribution for Wilks's likelihood ratio criterion is also developed.

K. J. Arrow (Chicago, Ill.).

Vajda, S. An outline of the theory of the 'analysis of variance.' J. Inst. Actuaries Students' Soc. 7, 235–252 (1948).

Expository article.

Samuelson, Paul A. Exact distribution of continuous variables in sequential analysis. Econometrica 16, 191–198 (1948).

A brief but novel treatment of sequential testing in terms of Fredholm equations. Let z be the logarithm of the likelihood ratio for a single observation under two alternative hypotheses and let its density be $F(z)$; let $z_1, z_2, \cdots$ represent a sequence of observations and $Z_1, Z_2, \cdots$ its partial sums. A sequential test is terminated when first $Z_n < b$ or $Z_n > a$. The conditional density of Z_i is

$$F_i(Z_i) = \int_b^a F(Z_i - s) F_{i-1}(s) ds, \qquad i > 1,$$

with $F_1(s) = F(s)$. Define $g(Z) = \sum_1^\infty F_i(Z)$; then some typical results are: (1) $g(Z) = F(Z) + \int_b^a F(Z - s) g(s) ds$, (2) $\int_a^\infty g(Z) dZ$ is the power function, (3) $\int_{-\infty}^b g(Z) dZ + \int_a^\infty g(Z) dZ = 1$, (4) the expected sample size is $\int_{-\infty}^\infty g(Z) dZ$.

A. M. Mood.

Obukhov, V. M. Applicability of test figures. Akad. Nauk SSSR. Prikl. Mat. Meh. 11, 485–488 (1947). (Russian. English summary)

This is a posthumously published fragment from a larger work. The author considers the following problem. Let X_i denote the elements of a set of m objects and let $P(X_i)$ denote the number got by measuring a certain physical characteristic of X_i by a process P. Let $S_n = (X_1, \cdots, X_n)$ be a random sample of the X_i of size n (small in comparison with m). Let P_1 be a (possibly biased) process and P_2 a standard process, regarded as accurate and unbiased, for measuring the same quantity. Let X_0 be the "true mean" of the quantity measured by the P_1 and P_2 and put $x_i = P_1(X_i)$, $y_i = P_2(X_i)$, $x_0 = m^{-1} \sum_{i=1}^m x_i$, $\bar{x} = n^{-1} \sum_{i=1}^n x_i$, $\bar{y} = n^{-1} \sum_{i=1}^n y_i$. Three estimates of X_0 are considered: $q_1 = x_0 + \bar{y} - \bar{x}$; $q_2 = \bar{y}$; $q_3 = a + bx_0$, where $y = a + bx$ is the regression line of the y's on the x's. It is argued that q_3 has a smaller sample variance than either q_1 or q_2 and that, as between q_1 and q_2, q_1 is to be preferred to q_2 (or q_2 to q_1) if $K = \sigma^2(q_1)/\sigma^2(q_2) = 1 + \lambda^2 - 2\lambda r_{xy} < 1$ (>1), where $\lambda = \sigma_x/\sigma_y$ and r_{xy} is the correlation coefficient of x and y. No assumptions as to the distribution of the $P(X_i)$ are stated. The English summary is unenlightening.

A. A. Brown (Cambridge, Mass.).

Schützenberger, M. P. Étude statistique d'un problème de sociométrie. Gallica Biologica Acta 1, 9 pp. (1948).

The following investigation is described. Each member of a group was asked to designate several people in the group who were (a) his best friends, (b) most distasteful to him. The problem is to test the null hypothesis that the reciprocal designations occur entirely through the operation of chance.

J. Wolfowitz (New York, N. Y.).

Geppert, Maria Pia. Maximum-likelihood-Schätzung und Rückschlussverteilung. Z. Angew. Math. Mech. 28, 85–91 (1948).

The author's summary is as follows. It is shown that, in estimating a parameter of a population from a statistic in a sample, neither the mode, nor the mean value, nor the median of the fiducial distribution, need coincide with the maximum likelihood estimate.

J. Wolfowitz.

Smith, H. Fairfield. Standard errors of means in sampling surveys with two-stage sampling. J. Roy. Statist. Soc. (N.S.) 110, 257–259 (1947).

Wilcoxon, Frank. Probability tables for individual comparisons by ranking methods. Biometrics 3, 119–122 (1947).

Patnaik, P. B. The power function of the test for the difference between two proportions in a 2×2 table. Biometrika 35, 157–175 (1948).

The paper gives several approximations to the power function of essentially the classical test of: (i) the hypothesis that two binomial parameters (p_1, p_2) are equal (against alternatives $p_1 < p_2$, $p_1 > p_2$); (ii) the hypothesis that $p_1 \leq p_2$ (against alternatives $p_1 > p_2$). Tables are given for approximating the power when p_1, p_2, the significance level (α), and the sample sizes are given; and for approximating the sample sizes required for given power when p_1, p_2, α, and the ratio of sample sizes are given.

Let p_j be the probability associated with an attribute A in a binomial population Π_j ($j = 1, 2$). Let the results of random samples S_1, S_2, from Π_1, Π_2, respectively, be presented in a 2×2 table:

	A	not A	Total
S_1	a	c	m
S_2	b	d	n
Total	r	s	N

The statistic $u = [a - (rm)/N]\{(mnrs)/[N^2(N-1)]\}^{-\frac{1}{2}}$ is used by the author to test (i) and (ii). The simplest approximation obtained for the power function of the u test of (i) is $P = 1 - \Phi(u_{\alpha/2} - h(\mu_1)) + \Phi(-u_{\alpha/2} - h(\mu_1))$, where $\Phi(x)$ is the standard normal distribution, $\Phi(u_{\alpha/2}) = 1 - \alpha/2$, $\mu_1 = mp_1 + np_2$, and

$$h(\mu_1) = \{mn\mu_1(N - \mu_1)/N^2(N-1)\}^{\frac{1}{2}} \log_e p_1(1 - p_2)/p_2(1 - p_1).$$

A similar result holds for the u test of (ii). The error of the approximations is shown for special cases. The following quantities are tabulated: $P = P(h, \alpha)$ for $\alpha = .10, .05, .02, .01$ and $h = .1(.1)3.0(.2)5.0$, where $h = |h(\mu_1)|$; $k(p_1, p_2) = h'/n^{\frac{1}{2}}$ for $p_1, p_2 = .05(.05).95$, where h' is defined only for $m = n$, and $h' \doteq h$; the ratio $h/n^{\frac{1}{2}}$ for $h = .1(.1)3.0(.2)5.0$ and $n = 10(5)50(10)100, 150$. *D. F. Votaw, Jr.*

Bateman, G. On the power function of the longest run as a test for randomness in a sequence of alternatives. Biometrika 35, 97–112 (1948).

Let $X_1, \cdots, X_{r_1 + r_2}$ be chance variables which can take only the values 0 and 1. All distributions in this paper are conditional, under the assumption that there are r_1 elements one and r_2 elements zero. The hypothesis H_0 states that the X's are identically and independently distributed; the hypothesis H_1 states that they constitute a simple Markov chain; the hypothesis H_2 states that the X's constitute a Markov chain of order two. Let g be the length of the run of greatest length and T be the number of runs. The author obtains the distributions of g and T (separately) under H_0, H_1 and H_2. On the basis of computations for a number of

values of the parameters he concludes that the test based on T is more powerful than the test based on g for discriminating between H_0 and H_1. Similar comparisons as regards power to discriminate between H_0 and H_2 have not yet led to conclusive results. *J. Wolfowitz.*

Quenouille, M. H. A large-sample test for the goodness of fit of autoregressive schemes. J. Roy. Statist. Soc. (N.S.) 110, 123–129 (1947).

In the analysis of stationary time-series, much attention has been paid in recent years to the hypothesis of linear autoregression, say (*) $x_i + a_1 x_{i-1} + \cdots + a_k x_{i-k} = u_i$, where $x_1, \cdots, x_n$ is the given time-series and $\rho_k(u) = E(u_i u_{i \pm k}) = 0$ for $k \neq 0$. Writing $r_k = \text{cov}(x_i, x_{i \pm k})$ and $\rho_k = \rho_k(x)$ for the observed and the hypothetical correlograms, an adequate test of the hypothesis (*) should amount to a simultaneous comparison between the r_k and the ρ_k, but such a test has not been available. Now the author develops a large sample test of this type. Defining A_i by $(1 + a_1 x + \cdots + a_k x^k)^2 = \sum A_i x^i$, forming $R_q = \sum A_i r_{q-i}$ ($q = k+1, k+2, \cdots$; i runs from $-\infty$ to $+\infty$ in both summations), and supposing the u_i to be normally distributed, it is shown that, for large n, (1) the forms R_q are independently and normally distributed about zero, with variances independent of q; (2) the joint distribution of $r_p - \rho_p$ ($p = 1, \cdots, k$) is independent of the R_q. Thus, using the equations $r_p = \rho_p$ to fit the scheme (*), we can for any m enter the χ_m^2-table with $R_{k+1}^2 + \cdots + R_{k+m}^2$ to test the adequacy of the fit.

The author applies his test to various instances of autoregression in earlier literature. It gives results in agreement with theory when applied to artificial time-series x_i constructed according to (*), but when examining the autoregressions fitted to 4 different empirical series, his test rejects the hypothesis in 3 cases, thus being more searching than earlier tests. Finally, the author shows how his test may be modified to account for a superimposed variation, say $x_i = x_i^* + v_i$, where x_i^* follows (*), the v_i are independent of the x_i^* and $\rho_k(v) = 0$ for $k \neq 0$. *H. Wold* (Uppsala).

Blake, Archie. Criteria for the reality of apparent periodicities and other regularities. Publ. Bureau Central Séismologique Internat. Sér. A. no. 16, 5 pp. (1946).

The paper contains comments of a general nature.
H. Wold (Uppsala).

Kempthorne, O. The factorial approach to the weighing problem. Ann. Math. Statistics 19, 238–245 (1948).

The author gives a short but lucid exposition of the technique of designing factorial experiments in fractional replications. He remarks that this technique is eminently suited for efficient weighing of objects on a chemical scale, since interactions may be confidently assumed to be absent. Thus the various aliases of the main effects may be identified without danger of error. The proposed technique is exemplified by an example of ten weighings and its efficiency discussed and compared with a procedure proposed by A. M. Mood [same Ann. 17, 432–446 (1946); these Rev. 8, 478]. *H. B. Mann* (Columbus, Ohio).

Bose, R. C. Recent work on "incomplete block designs" in India. Biometrics 3, 176–178 (1947).

Mathematical Biology

Wiener, Norbert, and Rosenblueth, Arturo. The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle. Arch. Inst. Cardiol. México 16, 205–265 (1946). (English. Spanish summary)

This study consists of two fairly distinct parts. The first, by far the simpler, and the more complete, is concerned with flutter, which is characterized by regular periodicities. Fibrillation, on the other hand, is characterized by lack of regularity, and in the second part a basis for a statistical approach is developed.

Regarding the heart as a syncytium of anastomosing fibers that may comprise virtually a two-dimensional or a three-dimensional system, and taking account of the finite refractory period, the authors consider the conditions necessary for the support of a persisting circulating wave of activity. The wave is propagated according to Huyghens' principle, except that the propagation proceeds only forward into the resting (not into the refractory) tissue. For one-dimensional and two-dimensional systems it is shown, for three-dimensional systems it is suggested, "that for enduring flutter there must always exist at least one non self-intersecting closed path which is the shortest of all paths topologically equivalent to it and which is of length not less than a wave-length." By wave-length here is meant the distance from wave-front to wave-back.

For fibrillation the authors are content to set up a statistical model, along the lines of Wiener and Wintner [Amer. J. Math. 65, 279–298 (1943); these Rev. 4, 220]. First the anastomoses are supposed to be distributed randomly in space according to a Poisson distribution. No choice is made of assumptions as to their connections. Next the authors turn to the distribution in time of the excitations at a single node, discussed in terms of the probability $p_{n_1 \cdots n_k}^{i_1 \cdots i_k}$ that on intervals I_j there shall be n_j active instants. A generating function is introduced and its derivatives, evaluated for fixed values of the variables, shown to be positive additive functionals of the sets I_j, hence representable as generalized Stieltjes integrals. By subdividing the intervals I_j into subintervals of lengths less than a refractory period so as to obtain multilinear forms, the generating function f is obtained as a polynomial with coefficients expressed in terms of iterated Stieltjes integrals. A similar representation expresses the distribution of active instants on the time lines through all the nodes. The authors leave the problem after expressing the laws of excitation in terms of this functional.
                                  A. S. Householder (Oak Ridge, Tenn.).

Selfridge, Oliver. Studies on flutter and fibrillation. V. Some notes on the theory of flutter. Arch. Inst. Cardiol. México 18, 177–187 (1948).

This paper completes and corrects in some details the study of flutter by Wiener and Rosenblueth [cf. the preceding review]. The correction consists in demonstrating the possibility of flutter in a region without obstacles, the flutter being, however, unstable. The completion consists in demonstrating the possibility of stable periodicity in any region possessing a finite number of nonequivalent closed paths.   *A. S. Householder* (Oak Ridge, Tenn.).

De Donder, Th. Le calcul des variations introduit dans la théorie des espèces et des variétés. III. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 34, 122–125 (1948).

In this note the problems of the stability of natural laws, and of hereditary phenomena, are related to the variational

formulas introduced in parts I and II [same Bull. Cl. Sci. (5) 33, 502–506, 718–724 (1947); these Rev. 9, 297].
                                  A. S. Householder (Oak Ridge, Tenn.).

Rashevsky, N. Development of mathematical biophysics in U. S. A. from 1939 to 1945 inclusive. Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 14, 28 pp. (1946).

Mathematical Economics

Leontief, Wassily. Introduction to a theory of the internal structure of functional relationships. Econometrica 15, 361–373 (1947).

A problem of growing importance in economics is, given a general functional relationship such as $y = F(x_1, \cdots, x_n)$, is it possible, on the basis of the mathematical properties of this function, to establish the existence of meaningful subgroups of variables and describe the properties of the corresponding partial functions? The author gives a possible mathematical solution in terms of the elementary theory of sets. Most of the definitions and propositions on functional separability, etc., required for this solution have been presented elsewhere by the author [Bull. Amer. Math. Soc. 53, 343–350 (1947); these Rev. 8, 452]. The author discusses the application of the method to economic analysis, and in particular to the theory of consumers' choice. It is suggested that the first step toward effective statistical application is the development of the basic concepts in terms of stochastic analysis. For this purpose a redefinition of the sets will be necessary.   *M. P. Stoltz* (Providence, R. I.).

Eyraud, Henri. De quelques problèmes d'économie pure. Ann. Univ. Lyon. Sect. A. (3) 10, 75–88 (1947).

This paper consists of a series of theorems concerning price determination under conditions of competition and various forms of monopoly, the incidence of taxation and the tariff. These theorems are based on highly simplified assumptions and are generalized from linear demand and cost functions. While formally valid, the stated theorems are not meaningful. Elasticity of demand is identified with the slope of the demand curve.   *M. P. Stoltz.*

Gheorghiu, Şerban. Sur la théorie de l'équilibre économique. Acad. Roum. Bull. Sect. Sci. 26, 377–381 (1946).

The author gives definitions and necessary and sufficient conditions for equilibrium of exchange based on the Walras-Pareto theory; they were developed in detail in a previous article [Bull. École Polytech. Bucarest [Bul. Politechn. București] 15, 22–60 (1944); these Rev. 7, 214]. The definitions of equilibrium conditions, and hence the stability conditions, are highly specialized. Thus for an open economy it is required that each exchangist exactly balance his consumption or production in a given time period with a corresponding increment to his stock by exchange and that prices do not change. On these conditions it is concluded that economic stability is impossible without stable money.
                                  M. P. Stoltz (Providence, R. I.).

TOPOLOGY

Besicovitch, A. S. On Crum's problem. J. London Math. Soc. **22** (1947), 285–287 (1948).

It is well known [see, e.g., W. W. R. Ball, Mathematical Recreations and Essays, 11th ed., Macmillan, London, 1939; New York, 1947, p. 224; these Rev. **8**, 440] that the Euclidean plane admits a set of four regions, but no set of five, all touching one another in the map-coloring sense. Moreover, the four regions can be convex polygons, such as a triangle surrounded by three trapezoids. The author proves that the analogous number of regions in 3-space is infinite: he constructs an infinite sequence of convex polyhedra such that every two have a common face. However, he does not say whether it is possible to find more than five such polyhedra with the further property that every three have a common edge. *H. S. M. Coxeter* (Toronto, Ont.).

Rado, R. A sequence of polyhedra having intersections of specified dimensions. J. London Math. Soc. **22** (1947), 287–289 (1948).

The author generalizes Besicovitch's result [see the preceding review] by finding in Euclidean n-space an infinite sequence of convex polytopes such that every k of them have a common $(n-k+1)$-dimensional element for each value of k up to $[\frac{1}{2}(n+1)]$. He then describes a set of $n+2$ convex polytopes having this property for each value of k up to n itself. This figure can be constructed by taking two homothetic simplexes, one inside the other, and joining corresponding cells by pyramidal frusta of type $(n|1)$ in the notation of D. M. Y. Sommerville [An Introduction to the Geometry of n Dimensions, Methuen, London, 1929, p. 103]. In other words, the large simplex is dissected into $n+2$ convex pieces: the small simplex and $n+1$ frusta. The author conjectures that there is no set of more than $n+2$ such polytopes. *H. S. M. Coxeter* (Toronto, Ont.).

Doss, Raouf. Sur la théorie de l'écart abstrait de M. Fréchet. Bull. Sci. Math. (2) **71**, 110–122 (1947).

This paper amplifies two previous notes [C. R. Acad. Sci. Paris **223**, 14–16, 1087–1088 (1946); these Rev. **8**, 48, 285]. An abstract écart E is a linearly ordered family of binary relations on a topological space X such that the sets xR form a basis for X with x ranging in X and R in E. The écart E is symmetric if each R is symmetric, and locally regular if given x and R there is an S such that xSS is contained in xR. The main result is that either X is metrizable (in fact, if and only if E has a countable cofinal subset) or one can construct a uniformly equivalent abstract écart F each relation of which is idempotent, and which is thus uniformly regular. In the nonmetric case, X is thus 0-dimensional. Other characterizations of regularly écartized spaces are given: (a) in terms of coverings, (b) in terms of the existence of uniformly continuous functionals. *R. Arens* (Los Angeles, Calif.).

Fréchet, Maurice. Sur les espaces à écart régulier et symétrique. Bol. Soc. Portuguesa Mat. Sér. A. **1**, 25–28 (1948).

Two results of R. Doss about spaces having a locally regular, symmetric abstract écart, are established without explicit use of Doss's main theorem [cf. the preceding review]. (a) For any open set V and point x within V there is a uniformly continuous functional f vanishing without V and having the value 1 at x; (b) the spaces in question are completely normal. Proposition (a) is part of a more general characterization of Doss's; and (b) strengthens an observation of the reviewer's [see the review of the author's paper in Portugaliae Math. **5**, 121–131 (1946); these Rev. **8**, 48]. *R. Arens* (Los Angeles, Calif.).

Finzi, A. Su una questione posta da S. Lie. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 185–188 (1947).

Soient V un espace métrique compact de dimension finie, et G le groupe des homéomorphismes de V sur V, muni de la topologie de la convergence uniforme sur V. Pour qu'une $T \varepsilon G$ appartienne à un sous-groupe à un paramètre de G, il faut que, si une variété $V' \subset V$ est invariante par T^n sans l'être par T, on puisse plonger V' dans une famille de ∞^1 variétés homéomorphes à V' et invariantes par T^n. L'auteur affirme que les $T \varepsilon G$ qui n'appartiennent à aucun sous-groupe à un paramètre de G forment un sous-ensemble partout dense dans G; dans le cas où V est le tore à une dimension, l'auteur montre par un exemple que ce sous-ensemble possède même des points intérieurs. *R. Godement.*

Samuel, P. On universal mappings and free topological groups. Bull. Amer. Math. Soc. **54**, 591–598 (1948).

Given a set E with a structure S, and the appropriate mappings $\phi: E \to F$ of E into sets F with another (and usually more elaborate) structure T, the problem of "universal mappings" consists in finding a particular set F_0 and a particular mapping $\phi_0: E \to F_0$ such that every $\phi: E \to F$ can be represented as a composite $\phi = f\phi_0$ for a suitable mapping $f: F_0 \to F$. The author shows that this problem always has a solution, provided that the mappings to be considered satisfy certain natural axioms, including especially the existence of, and mappings for, Cartesian products, and a limitation on the cardinal number of the "closure" of a subset of a set F. In particular, the author must prove [p. 593] that a mapping $\phi: E \to F_1$ can be restricted to the closure F_0 of $\phi(E)$ in F_1; this appears to require either the consideration of mappings $F_1 \to F_0$ not everywhere defined, or the addition of an axiom $(S-T)_3$ similar to his axiom I_2. Numerous examples (Čech compactification, completion of a metric space, embedding problem, etc.) are noted. The chief application is a treatment of a universal mapping of a topological space into a topological group, as in Markoff's theory of free topological groups [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **9**, 3–64 (1945); these Rev. **7**, 7]. This treatment is similar to that previously given by Kakutani [Proc. Imp. Acad. Tokyo **20**, 595–598 (1944); these Rev. **7**, 240]. *S. Mac Lane* (Chicago, Ill.).

Chern, Shiing-shen, and Jou, Yuh-lin. On the orientability of differentiable manifolds. Sci. Rep. Nat. Tsing Hua Univ. **5**, 13–17 (1948).

The Grassmann manifold of all n-planes through the origin in real m-space is orientable if and only if m is even. In real projective n-space, the hyperquadric $\pm x_1^2 \pm \cdots \pm x_{n-1}^2 - x_n x_{n+1} = 0$ is orientable if and only if n is odd or the hyperquadric is a hypersphere (all $\pm$ signs above are $+$). *H. Whitney* (Cambridge, Mass.).

Higman, G. A theorem on linkages. Quart. J. Math., Oxford Ser. 19, 117–122 (1948).

Let K be a polygonal linkage in the three-sphere S^3. It is shown that if the fundamental group $\pi_1(S^3-K)$ is a non-trivial free product of two groups then the second homotopy group $\pi_2(S^3-K)$ is nontrivial. The argument is almost entirely algebraic and starts out with the equations by means of which J. H. C. Whitehead [Fund. Math. 32, 149–166 (1939)] gave a description of the group $\pi_2(S^3-K)$. The work casts light on the interesting possibilities of Whitehead's equations.   *S. Eilenberg.*

Hurewicz, W., Dugundji, J., and Dowker, C. H. Continuous connectivity groups in terms of limit groups. Ann. of Math. (2) 49, 391–406 (1948).

Les auteurs développent la théorie d'homologie "singulière" en termes de limites directes et inverses. Ils partent algébriquement d'un ensemble Ω partiellement ordonné et d'un système de groupes $\{G_\sigma\}$ $(\sigma \epsilon \Omega)$ et d'homéomorphismes $h_{\rho\sigma}^i$ $(\rho<\sigma)$ de G_σ en G_ρ, où ils supposent les propriétés usuelles, mais où ils admettent que l'indice i a un nombre arbitraire de valeurs. Le groupe-limite de $\{G_\sigma\}$ est définie de la manière ordinaire. Topologiquement, on introduit des polytopes P représentés (f) en Y et on définit l'ordre partiel $(f,P)<(f_1,P_1)$ s'il existe une représentation isomorphe de P en P_1 tel que $f_1\varphi=f$. Les groupes d'homologie et de cohomologie des P engendrent ceux de Y (comme des limites inverses et directes), entre lesquels il y a une dualité. (Pour éviter des paradoxes ensemblistes on doit borner la puissance des polytopes P; il y a des notions qui dépendent de cette borne.) Comme application on peut généraliser un théorème de W. Hurewicz sur les polytopes asphériques [Nederl. Akad. Wetensch., Proc. 39, 215–224 (1936)]. Au moyen de recouvrements de Y on peut introduire: (1) les groupes de Čech basés sur les nerfs; (2) les groupes d'Alexander, duaux des groupes de Vietoris. Si chaque recouvrement possède des raffinements barycentriques, ils sont isomorphes.   *H. Freudenthal* (Utrecht).

Pontryagin, L. S. The general topological theorem of duality for closed sets. Uspehi Matem. Nauk (N.S.) 2, no. 2 (18), 45–55 (1947). (Russian)

Translated from Ann. of Math. (2) 35, 904–914 (1934).

Stinrod [Steenrod], N. E. Regular cycles of compact metric spaces. Uspehi Matem. Nauk (N.S.) 2, no. 2(18), 56–78 (1947). (Russian)

Translated from Ann. of Math. (2) 41, 833–851 (1940); these Rev. 2, 73.

Lerĕĭ, Ž. [Leray, J.], and Šauder, Yu. [Schauder, J.]. Topology and functional equations. Uspehi Matem. Nauk (N.S.) 1, no. 3–4 (13–14), 71–95 (1946). (Russian)

Translated from Ann. Sci. École Norm. Sup. (3) 51, 45–78 (1934).

Kaplan, Wilfred. Topology of level curves of harmonic functions. Trans. Amer. Math. Soc. 63, 514–522 (1948).

Let F be a regular curve family filling the (x, y)-plane (a family locally homeomorphic with a family of parallel lines). It is shown that there exists a homeomorphism of the plane onto a domain D such that F is transformed into the family of level curves of a harmonic function. Say that F is hyperbolic or parabolic if D can be chosen as the interior of the unit circle or as the whole plane, respectively. Every F is hyperbolic; some, but not all, are also parabolic. A theorem concerning the subdivision of the Riemann surface of the inverse of a function analytic in a simply-connected domain is given. The proofs are based on previous results of the author on families of curves filling the plane [Duke Math. J. 7, 154–185 (1940); 8, 11–46 (1941); Lectures in Topology, University of Michigan Press, Ann Arbor, Mich., 1941, pp. 299–301; these Rev. 2, 322; 3, 135]. The possibility of alternate developments is discussed briefly.
             H. Whitney (Cambridge, Mass.).

Hamilton, O. H. Fixed point theorems for interior transformations. Bull. Amer. Math. Soc. 54, 383–385 (1948).

Let M be a bounded locally connected plane continuum which does not separate the plane. A continuous interior transformation of M onto a topological 2-cell in the plane containing M always has a fixed point. Similarly a continuous interior transformation of a 2-cell I onto any continuum in the plane containing I has a fixed point.
             S. Eilenberg (New York, N. Y.).

Jones, F. Burton. Concerning non-aposyndetic continua. Amer. J. Math. 70, 403–413 (1948).

The author is concerned with internal properties of continua that are not conditioned locally. Let M be a compact metric connected space. If x, y are distinct points of M then M is aposyndetic at x with respect to y if there exist a continuum H and an open set U such that $x\epsilon U \subset H \subset M-y$. Further, M is aposyndetic at x if it is aposyndetic at x with respect to each $y\epsilon M-x$. A point p of M is a weak cut point if there exist points $x, y\epsilon M-p$ such that every continuum containing both x and y also contains p. The first section of the paper is devoted to continua that are not aposyndetic at some points. It is shown that the Cartesian product of two nondegenerate continua is aposyndetic at every point. The next section concerns indecomposable continua and in the third section it is shown that a continuum aposyndetic at no point must contain at least one weak cut point and may contain no more such points. The paper terminates with results on continua containing only one weak cut point or containing no weak cut point. It should be remarked that every point of an indecomposable continuum is a weak cut point so that the author's theorems high-light some of the peculiar properties of continua.
             A. D. Wallace (New Orleans, La.).

GEOMETRY

Forder, H. G. On the axioms of congruence in semi-quadratic geometry. J. London Math. Soc. 22 (1947), 268–275 (1948).

The paper shows that the fifth of a set of five axioms used by R. L. Moore [Trans. Amer. Math. Soc. 9, 487–512 (1908)] as a basis for congruence in elementary geometry can be derived from the first four and two of the special cases of the fifth axiom, assuming order axioms but making no assumptions concerning continuity, parallels, or intersections of circles with circles or lines. It is stated that the adjunction of the Euclidean parallel axiom permits proof of the Pythagorean theorem and an interval of length $(a^2+b^2)^{\frac{1}{2}}$ may always be constructed from intervals of lengths a, b. Since, on the other hand, it is not always possible to construct $(a^2-b^2)^{\frac{1}{2}}$, $a>b$, the author refers to the geometry as semi-quadratic.   *L. M. Blumenthal* (Columbia, Mo.).

Yang, Chung-Tao. Certain chains in a finite projective geometry. Duke Math. J. 15, 37–47 (1948).

In einer endlichen ebenen projektiven Geometrie über einem Galoisfeld der Ordnung p^n werden die Gruppen von p^r+1 Punkten (sog. r-fache Ketten) für $r=2^m$ untersucht, die in einer projektiven Unterebene enthalten sind. Nur wenn 2^m ein Teiler von n ist, existieren solche Ketten. Die Begriffsbildungen, zu denen sie Anlass geben und ihre Eigenschaften sind vielfach analog zu den Eigenschaften der Ketten, die von Staudt 1847 untersucht hat. Eine ebene Collineation, die eine 2^m-fache Unterkette C_1 einer 2^{m+1}-fachen Kette C_2 invariant lässt, teilt die nicht zu C_1 gehörigen Punkte von C_2 in Paare conjugierter Punkte. Näher untersucht werden simultane 2^m-fache Ketten, von denen je zwei orthogonal sind, d.h. von denen eine ein Paar conjugierter Punkte der anderen enthält. *R. Moufang.*

Wiman, A. Über der Hesseschen Konfiguration in der Ebene entsprechende Konfigurationen in höheren Räumen. Ark. Mat. Astr. Fys. 34A, no. 18, 19 pp. (1948).

The three most important finite ternary collineation groups are the simple groups of orders 168 and 360 and a group G_{216} of order 216 which C. Jordan has called the Hessian group. This paper is devoted to the derivation and study of a collineation group $G_{p^3(p^2-1)}$ on the variables x_i ($i=0, 1, \cdots, p-1$) (mod p), in p dimensions (where $p=2q+1$ is a prime), which generalizes the Hessian group ($p=3$). First a normal subgroup G_{p^2} of the type (p, p) is described which contains $p+1$ cyclic subgroups generated respectively by $S: (x_i'=\epsilon^i x_i)$, and $S^h T: (x_i'=\epsilon^{hi}x_{i+1}$, $h=0, 1, \cdots, p-1)$, where ϵ is a primitive pth root of unity. Next a set of $p+1$ polytopes P_{-1} and P_h ($h=0, 1, \cdots, p-1$) are each defined by selecting as vertices the p fixed points under the corresponding subgroup. This generalizes the Hessian configuration of 4 triangles, the sides of each pair of which intersect in the same 9 points (to be called M_ϵ's) and for which the 9 joins of vertices for any pair are the same 9 lines (to be called $M_{\epsilon+1}$'s). For P_{-1} the coordinate hyperplanes are faces, whereas for P_h the p hyperplane faces are $\sum_k \epsilon^{ik} x_{i+k}=0$, where j stands for $\frac{1}{2}(k^2+kp)$ and k is summed from 0 to $p-1$. A set of p^2 manifolds M_q of $q=\frac{1}{2}(p-1)$ dimensions are defined by $x_i=0=\epsilon^{-(k+\nu)h}x_{i-k}+\epsilon^{(k+\nu)h}x_{i+k}$ ($h=1, 2, \cdots, q$). These correspond to the 9 points when $p=3$. For fixed ν and k and variable i there are p manifolds M_q lying in the same k-face of the polytope P_ν. Another set of p^2 manifolds M_{q+1} defined by $\epsilon^{-(k+\nu)h}x_{i-k}=\epsilon^{(k+\nu)h}x_{i+k}$ ($h=1, 2, \cdots, q$) correspond to the 9 lines of the Hessian configuration. Adjoining an involution enlarges the G_{p^2} to a G_{2p^2}. Adjoining the substitution $x_k'=\epsilon^j x_k$ ($k=0, 1, \cdots, p-1$), where again j stands for $\frac{1}{2}(k^2+kp)$, extends the G_{2p^2} to a G_{2p^3}. The substitution $x_i'=x_{gi}$ ($i=0, 1, \cdots, p-1$) where g is a primitive residue (mod p) increases the order by a factor $q=\frac{1}{2}(p-1)$ and a group $G_{p^3(p-1)}$ is obtained. Finally the permutation of the $p+1$ polyhedrons among themselves by $x_i'=\rho \sum_k \epsilon^{ik} x_k$ extends the group order to $p^3(p^2-1)$. The paper concludes with a discussion of certain elliptic "normal curves" invariant under the subgroup G_{2p^3}. *J. S. Frame.*

Popa, Ilie. Sur un théorème de géométrie élémentaire. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 79–80 (1947).

The author offers one more proof and some discussion of the theorem: a triangle may be constructed with the distances of the vertices of an equilateral triangle from any point of its plane. Some of the references are not sufficiently detailed. *N. A. Court* (Norman, Okla.).

Thébault, Victor. On the twelve point sphere of the tetrahedron. Amer. Math. Monthly 55, 357 (1948).

Usai, Giuseppe. Rombi e quadrati in corrispondenza a punti di certe curve piane. Matematiche, Catania 2, 84–107 (1947).

Pachta, Zdeněk. The vertex as a base point of a pencil of conics. Časopis Pěst. Mat. Fys. 72, D74–D78 (1947). (Czech)

Piska, Rudolf. Contribution to the construction of the tangents and centers of curvature of certain bicircular quartics. Časopis Pěst. Mat. Fys. 72, D78–D83 (1947). (Czech)

Pettineo, B. Alcune proprietà delle omografie tra due piani. Matematiche, Catania 1, 212–216 (1946).

Convex Domains, Integral Geometry

Fejes, László. Über eine extremale Bedeckung des Raumes durch konvexe Polyeder. Mat. Fiz. Lapok 51, 19 pp. (1944). (Hungarian. German summary)

In the first two chapters the author discusses various extremal problems of the splitting of the plane into convex polygons and space into convex polyhedra. The problems in the plane are as follows. (1) Let us split the plane into convex polygons, each of which contains a circle of radius 1, and so that in a large circle the number of polygons shall be maximal. (2) Let us split the plane into polygons each of which is contained in a circle of radius 1 and so that the number of polygons in a large circle shall be minimal. (3) Split the plane into polygons of area 1 so that the sum of the lengths of the edges falling within a large circle shall be minimal. In all three cases the solution of the extremal problem is the regular hexagonal lattice. In space none of the three analogous problems is solved so far.

Next the author proves the following theorem. Split the plane into polygons so that each polygon contains a circle of fixed radius and is contained in a circle of fixed radius. Put $\lim \sup_{r\to\infty} K[k^2/t; r]=K(k^2/t)$, where $K[k^2/t; r]$ denotes the arithmetic mean of the square of the circumference divided by the area of the polygons falling in a circle of radius r (k denotes the circumference, t the area). The author proves that (1) $K(k^2/t)\geqq 8\sqrt{3}$. He also gives necessary and sufficient conditions that there should be equality in (1). (Equality occurs if and only if almost all polygons differ arbitrarily little from a regular hexagon of fixed size.)

The author further considers an analogous problem in space and proves the following theorem. Split space into convex polyhedra, each of which contains a sphere of fixed radius and is contained in a sphere of fixed radius. Let F, V and M denote the surface, volume and integral of the mean curvature of the polyhedra; let $K(F^2/V; R)$ and $K(M; R)$ denote the arithmetic means of F^2/V and M for all polyhedra contained in a sphere of radius R. Then (2) $\lim \sup_{R\to\infty} (K(F^2/V; R)/K(M; R))\geqq (6\sqrt{3})/\pi$. The case of equality is again completely determined. Various applications of (2) are given. *P. Erdős.*

Allendoerfer, Carl B. Steiner's formulae on a general S^{n+1}. Bull. Amer. Math. Soc. 54, 128–135 (1948).

Let V^n be a hypersurface in a complete Riemann space S^{n+1} of constant curvature K which is bounding in S^{n+1}.

The principal curvatures of V^n with respect to an outward normal are all assumed to be negative. Let $V_\rho{}^n$ be a surface parallel to V^n at a distance ρ measured along outwardly drawn geodesics with $0 \leq \rho \leq \pi/2K^{\frac12}$ for $K > 0$, and $\rho \geq 0$ for $K < 0$. The author develops explicit formulas for the area A_ρ of $V_\rho{}^n$ and the volume Vol_ρ bounded by it which are generalizations of the formulas of Steiner for plane curves and for surfaces in three-dimensional space; these formulas give the area and volume of $V_\rho{}^n$ in terms of K, ρ, and integrals of certain curvatures of V^n taken over V^n. Instead of giving the general formulas we reproduce the formulas for the special case $n = 2$, in which they can be simplified somewhat through the use of the Gauss-Bonnet formula developed by the author and A. Weil [Trans. Amer. Math. Soc. 53, 101–129 (1943); these Rev. 4, 169]:

$$A_\rho = A_0 + M_1 K^{-\frac12} \sin(\rho K^{\frac12}) \cos(\rho K^{\frac12}) + 2\pi\chi K^{-1} \sin^2(\rho K^{\frac12}),$$

$$\mathrm{Vol}_\rho = \mathrm{Vol}_0 + A_\rho + M_1(2K)^{-1}\sin^2(\rho K^{\frac12})$$
$$+ \pi\chi K^{-1}[\rho - K^{\frac12}\sin(\rho K^{\frac12})\cos(\rho K^{\frac12})].$$

In these formulas χ is the Euler characteristic of V_2 and M_1 is the integral over V_2 of the "mean curvature" of V_2; the formulas hold for $K < 0$ as well as for $K > 0$. The author states that his methods for deriving his formulas are similar to those used by Vidal Abascal for the case $n = 1$ [cf. the following review].   *J. J. Stoker* (New York, N. Y.).

Vidal Abascal, E. **Extension of the concept of parallel curves on a surface. Length and area corresponding to the curve thus obtained from another given one.** Revista Mat. Hisp.-Amer. (4) **7**, 269–278 (1947). (Spanish)

Vidal Abascal, E. **A generalization of Steiner's formulae.** Bull. Amer. Math. Soc. **53**, 841–844 (1947).

Vidal Abascal, E. **Area generated on a surface by an arc of a geodesic when one of its ends describes a fixed curve and length of the curve described by the other end.** Revista Mat. Hisp.-Amer. (4) **7**, 132–142 (1947). (Spanish)

The first paper contains the results of the last two as special cases. On a surface $x(u, v)$ let C be a simple closed curve bounding a domain of area F. Choose C as $u = 0$ and v, $0 \leq v \leq L$, as the arc length on C. Let the curve $v = v_0$ be a geodesic $g(v_0)$ which intersects C at the angle $w(v_0)$. Choose u as arc length on $v = $ constant, so that $g(0, v) = 1$, $E = 1$. Denote by C_ρ the curve $(\rho(v), v)$, $0 \leq v \leq L$. Then the arc bounded by C_ρ for sufficiently small $\rho(v)$ is

$$F_\rho = F + \int_0^L \sin w(v) \sin\{\rho(v)K^{\frac12}(0, v)\}K^{-\frac12}(0, v)dv$$
$$- \int_0^L \{K_g(v) - w'(v)\}K^{-1}(0, v)\{\cos[\rho(v)]K^{\frac12}(0, v) - 1\}dv$$
$$+ \frac{1}{4!}\int_0^L \frac{\partial K(0, v)}{\partial u}\rho^4(v)\sin[w(v)]dv + \cdots.$$

Here $K(u, v)$ is the Gauss (or total) curvature of $x(u, v)$ at (u, v) and $K_g(v)$ is the geodesic curvature of C at $(0, v)$. (If $K < 0$, the expression $\sin \rho K^{\frac12}$ means $\sinh \rho(-K)^{\frac12}$, and $\sin w(v)$ means $\sinh w(v)$, $\cdots$.) It is assumed that $K(u, v)$ is different from 0. If K does not depend on u the last term drops out and what remains is an exact expression for F_ρ.

If ρ is constant and $\alpha(v)$ denotes the angle of C_ρ and $g(v)$ then the length L_ρ of C_ρ is

$$L_\rho = \int_0^L \sin w(v)\cos\{\rho K^{\frac12}(0, v)\}\csc\alpha(v)dv$$
$$+ \int_0^L \{K_g(v) - w'(v)\}\sin\{\rho K^{\frac12}(0, v)\}K^{-\frac12}(0, v)\csc\alpha(v)dv$$
$$- \frac{\rho^3}{3!}\int_0^L \frac{\partial K(0, v)}{\partial u}\sin w(v)\csc\alpha(v)dv + \cdots.$$

Again, if K does not depend on u, the first two terms on the right give an exact expression for L_ρ. These formulas contain the analogues to Steiner's formulas for length and area of parallel curves on arbitrary surfaces, and in particular on surfaces with constant curvature. The latter case has recently been discussed by others; for the literature see Allendoerfer's paper reviewed above. Another consequence is the value

$$L_0 + [\delta t]_0^L - \int_0^L K_g \delta n\, ds - \frac{1}{2}\int_0^L (\delta n)^2 K ds$$
$$+ \frac{1}{3!}\int_0^L (\delta n)^3(K_g K^{-1} + K^{\frac12})ds + \cdots$$

for the length of a curve obtained from C by normal variation $-\delta n$ and tangential variation δt, with an obvious modification for $K < 0$.   *H. Busemann* (Los Angeles, Calif.).

Algebraic Geometry

Conforto, Fabio, e Zappa, Guido. **La geometria algebrica in Italia (dal 1939 a tutto il 1945).** Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 8, 43 pp. (1946).

Segre, Beniamino. **Geometria algebrica nei paesi anglosassoni (dal 1939 al 1945).** Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 11, 52 pp. (1946).

Segre, Beniamino. **Sul massimo numero di nodi delle superficie di dato ordine.** Boll. Un. Mat. Ital. (3) **2**, 204–212 (1947).

The purpose of this note is to provide an example contradicting Severi's result [see the following review] that the maximum number of nodes for a surface of order m in S_3 without multiple curves is $\binom{m+2}{3} - 4$. The surface considered is of the form

$$F \equiv \begin{vmatrix} f_{11} & \cdots & f_{1r} \\ \cdots & \cdots & \cdots \\ f_{r1} & \cdots & f_{rr} \end{vmatrix} = 0, \qquad f_{ij} = f_{ji},$$

where the f_{ij} are polynomials of order n in the coordinates; thus $m = rn$. It is shown that (i) such surfaces form an algebraic system of dimension $\binom{r+1}{2}\binom{n+3}{3} - r^2$; (ii) the general surface of the system has $\binom{r+1}{3}n^3$ ordinary nodes and no other singularity; the system is thus superabundant if $r \geq 2$, $n \geq 4$; (iii) by a suitable choice of the polynomials f_{ij}, namely $f_{ii} = f_0 + f_i$, $f_{ij} = f_0$ ($i \neq j$), where f_0 is arbitrary and each of $f_1, \cdots, f_r$ is the product of n linear factors, we can impose on a surface of the system $\frac{1}{6}rn^3(n-1)$ further nodes and no other singularity, which contradicts Severi's result if $r \geq 2$, $n > \frac12(r+1)$.   *P. Du Val* (Istanbul).

Severi, Francesco. Sul massimo numero di nodi di una superficie di dato ordine dello spazio ordinario o di una forma di un iperspazio. Ann. Mat. Pura Appl. (4) 25, 1–41 (1946).

The main purpose of this paper is the proof of the following theorem. If a surface of order m in S_3 has δ nodes [i.e., ordinary conical double points] and no other singularities, then $\delta \leq \delta_0 = \binom{m-1}{3} - 4$. [The limitation $m \geq 3$ is not stated but presumably intended, since for $m=2$, $\delta_0=0$.] In particular, the limit $\delta = \delta_0$ is attained for $m=4$, $m=5$ by the surfaces of Kummer and Togliatti [Vierteljsch. Naturforsch. Ges. Zürich 85 Beiblatt (Festschrift Rudolf Fueter), 127–132 (1940); these Rev. 3, 15].

The method of proof is along the following lines. (i) The base number ρ of the surface, the number ρ_0 of transcendental 2-cycles and the Zeuthen-Segre invariant I satisfy $\rho + \rho_0 = I + 2$, and I for the surfaces considered is a function of m only, while $\rho \geq \delta + 1$; thus a lower limit for ρ_0 for given m gives an upper limit for δ. (ii) It is assumed practically without proof that for a surface variable in a continuous system a diminution in ρ_0 leads to an at least equal diminution in the number of moduli on which the surface depends. (iii) The moduli are counted by showing that for the surface without singular points these are merely the projective invariants. This part of the proof occupies much of the paper, and in the course of it a number of very general lemmas are established, the chief being the two following (with rather similar results for forms in higher space). Two forms of S_r ($r \geq 3$) having not more than ∞^{r-3} ordinary double points, which are birationally equivalent, and of which one is of order $m > r+1$, are homographic. A general surface F of order $m \geq 4$ of S_3 has no plane section homographic to its general plane section. The surfaces F passing through a general plane curve C_0 of order m, and homographic to a given one and hence to each other, form a complete ∞^4 system of homologous surfaces (the plane of homology being that of C_0).

Finally the suggestion is offered that the maximum number of nodes of a form of order m of S_r is probably $\binom{m+r-1}{r} - r - 1$, which agrees with the known results 10, 15, respectively for $m=3$, $r=4$, 5.

[It should be noted that B. Segre [cf. the preceding review] has shewn by examples that the assumption (ii) and the main theorem are both false, and that if the required number is a cubic polynomial in m the coefficient of m^3 must be greater than $\frac{1}{6}$.] P. Du Val (Istanbul).

Piazzolla-Beloch, Margherita. Curve algebriche piane d'ordine $2n$, con due punti multipli all'infinito di molteplicità n (coniche generalizzate). Ann. Univ. Ferrara 6, 11 pp. (1947).

L'auteur étend aux courbes C d'ordre $2n$ ayant deux points n-uples à l'infini les propriétés diamétrales classiques des coniques. Le diamètre d'une direction, pris au sens de Newton, est en relation linéaire avec la pente de cette direction: il en resulte l'existence d'un point principal par où passent tous les diamètres. De plus, une direction et celle de son diamètre sont liées involutivement: il en résulte l'existence de diamètres conjugués et de deux diamètres principaux conjugués et orthogonaux. Les directions confondues avec leurs conjuguées sont celles des points à l'infini: il en résulte que dans le cas où ces points sont confondus (cas parabolique) tous les diamètres sont parallèles à la direction asymptotique, le point à l'infini étant point principal, et il n'y a qu'un seul diamètre principal. Dans

le cas où les points à l'infini sont les points cycliques, tous les diamètres conjugués sont orthogonaux et principaux.

L. Gauthier (Nancy).

Dalla Volta, V. Su alcuni tipi di quartiche piane. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 301–303 (1947).

If a plane elliptic quartic has two biflecnodes, the points of contact of the two proper tangents from either node are collinear with the other. This and similar results are exhibited as examples of the fact that, if two involutions g_2^1 and ϑ_2^1 on an elliptic curve are such that two (distinct) double points of g_2^1 are mates in ϑ_2^1, then the mate in g_2^1 of any double point of ϑ_2^1 is another double point of ϑ_2^1. The plane quartic with three biflecnodes is discussed as a limiting case. J. G. Semple (London).

Bogdan, C. P. Sur les cubiques de l'espace ordinaire et sur les surfaces développables de leurs tangentes. Revista Ştiinţifică "V. Adamachi" 33, 150 (1947).

Fano, G. Su alcuni lavori di W. L. Edge. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 179–185 (1947).

This note is essentially a study of the properties of a curve of order eight and genus three in S_6, whose prime sections are the sets of the bicanonical series, and of the scrolls of lines in ordinary space, of the same order and genus, whose Grassmannians are curves of this type. These scrolls include as a particular case the scrolls of trisecants of the Jacobian curve of a general net of quadric surfaces, which is the subject of the papers by Edge referred to in the title. J. A. Todd (Cambridge, England).

Büke, Macit. Les surfaces quartiques de Segre nonsingulières dans l'espace projectif à quatre dimensions. II. Rev. Fac. Sci. Univ. Istanbul (A) 12, 255–288 (1947).

[A preceding installment of this part appeared in the same vol., 164–189 (1947); these Rev. 9, 56.] This continues the study of the quartic surface Γ in four dimensions given by the intersection of two quadric primals f and ϕ, and containing 16 lines [cf. part I, same vol., 80–106 (1947); these Rev. 9, 56]. The reality of the points and the lines on Γ is studied. Real equations lead to real surfaces, or else to surfaces with no real points or only a finite number of singular real points: called respectively visible and invisible surfaces. On Γ a line is either real, or has just one real point, or has no real point. Six types of nonsingular surface exist, $\Gamma_1, \cdots, \Gamma_6$, containing respectively (16, 0, 0), (8, 0, 8), (0, 0, 16), (4, 4, 8), (0, 8, 8), (0, 16, 0) of these three sorts of line: e.g., Γ_4 has 4 real, 4 imaginary each with one real point, and 8 entirely imaginary, lines. Each type can be deformed really and continuously into the next by passing through a singular type Φ of surface with one real double point Q, either a node or isolated. On Φ four pairs of the 16 lines coincide into four lines passing through Q, leaving a residual eight lines which constitute a double four of lines in [4]. Also Q lies on the associated line e of this double four. By varying Q on a given e for a given double four, three cases emerge for which the above four lines through Q coincide in pairs. This leads to a further refinement shewing that the type of surface Γ_3 is really two types, one visible Γ_{3a} and one invisible Γ_{3b}. The rectangular scheme, previously introduced, exhibits the connectivity of the 16 lines by

means of a two-dimensional plane figure, shewing the three sorts of line for the six visible cases Γ_i. There are respectively 40, 12, 0, 4, 4, 8 tritangent planes for the six Γ_i, which nearly characterize the six cases, while Γ_4 and Γ_5 are definitely characterized by the number of real lines upon them, namely 4 and 0. The corresponding finite groups, formed by such interchanges among the lines as retain the character of the surface Γ_i for fixed i, are of respective orders 1920, 96, 192, 32, 32, 384, that of Γ_1 being the general group already discussed.

A topological passage is based on work of P. Du Val. The tangent plane at a point P of Γ cuts the pencil of quadrics $f+\lambda\phi$ in an ∞^1 of line pairs which form an involution, elliptic, hyperbolic or parabolic. This naturally defines three types of point P. The parabolic points are those and only those which lie upon the 16 lines; and these lines which meet in pairs at 40 points P_{xx} lying by fives upon a line, form a generalized polyhedron which breaks the surface up into curvilinear facets whose edges are portions of these lines. When real lines only are considered and real facets, the interior points of a facet are entirely elliptic or entirely hyperbolic. Cases Γ_3, Γ_5, Γ_6 which have no visible polyhedron are entirely hyperbolic, with possible isolated k-points which are parabolic: Γ_6 has two sheets, the five others are one-sheeted surfaces. The polyhedron for each Γ_i is considered, that of Γ_1 having 20 quadrangular and 16 pentagonal facets, meeting in fours alternately at a vertex, the 20 being hyperbolic and the 16 elliptic. The surface is one-sided, as also Γ_2 (which has quadrangles and hexagons), and Γ_4 (which behaves like Klein's tetrahedral one-sided surface); Γ_3 behaves like an anchor ring, while Γ_5 and each sheet of Γ_6 behave like spheres. Projection of the surface to a prime from a point P of the surface yields five types of real cubic surface in [3] from the six Γ_i, the cases Γ_3 and Γ_4 merging into one such cubic. *H. W. Turnbull* (St. Andrews).

Room, T. G. Matrices of integers associated with self-transformations of surfaces. Proc. Roy. Soc. London. Ser. A. 193, 25–43 (1948).

The author's summary is as follows. Let $c=(c_1, \cdots, c_r)$ be a set of curves forming a minimum base on a surface, which, under a self-transformation, $\mathcal{T}$, of the surface, transforms into a set $\mathcal{T}c$ expressible by the equivalences $\mathcal{T}c=Tc$, where T is a square matrix of integers. Further, let the numbers of common points of pairs of the curves, c_i, c_j be written as a symmetrical square matrix Γ. Then the matrix T satisfies the equation $T\Gamma T'=\Gamma$.

The significance of solutions of this equation for a given matrix Γ is discussed, and the following special surfaces are investigated: Surfaces, in particular quartic surfaces, with only two base curves. Self-transformations of these depend on the solutions of the Pell equation $u^2-kv^2=1$ (or 4). The quartic surface specialized only by being made to contain a twisted cubic curve. This surface has an involutory transformation determined by chords of the cubic, and has only one other rational curve on it, namely, the transform of the cubic. The appropriate Pell equation is $u^2-17v^2=4$. The quartic surface specialized only by being made to contain a line and a rational curve of order m to which the line is $(m-1)$-secant (for $m=1$ the surface is made to contain two skew lines). The surface has two infinite sequences of self-transformations, expressible in terms of two transformations $\mathcal{R}$ and $\mathcal{S}$, namely, a sequence of involutory transformations $\mathcal{R}\mathcal{S}^n$, and a sequence of non-involutory transformations $\mathcal{S}^n$. *P. Du Val* (Istanbul).

Bogdan, C. P. Sulle superficie F^{n^2} rappresentative delle curve C^n piane. Acad. Roum. Bull. Sect. Sci. 27, 593–594 (1947).

Une simple remarque sur la surface rationnelle F d'ordre n^2 de l'espace à $\frac{1}{2}n(n+3)$ dimensions qui est représentée sur le plan par le système linéaire de toutes les courbes de l'ordre n. La surface F contient ∞^3 courbes rationnelles normales de l'ordre n qui sont représentées par les droites du plan. Si on fait la projection de F de l'espace qui contient $k<n-1$ de ces courbes sur un espace dual on obtient une surface analogue de l'ordre $(n-k)^2$; ce qui est bien évident sur la représentation plane. *E. G. Togliatti* (Gênes).

Vaccaro, G. Le ipersuperficie d'ordine n con un punto $(n-2)$-plo. I. Genesi delle singolarità della varietà di diramazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 288–293 (1947).

Dans cette note, l'auteur étudie un S_{r-1} double obtenu par projection d'une V_{r-1}^n de S_r à partir d'un point O, $(n-2)$-uple pour V_{r-1}^n, et la V_{r-2} de diramation de S_{r-1}. Une condition nécessaire et suffisante pour que V_{r-2}^{2n-2} de S_{r-1} soit variété de diramation d'un S_{r-1} double est qu'il existe une V_{r-2}^{n-2} tangente à V_{r-2}^{2n-2} tout le long de sa section complète par une V_{r-1}^{n-1}. Lorsque V_{r-1}^n admet un point double P, sans que OP lui appartienne tout entière, la projection P_1 de P est un point au moins double de la V_{r-2} de diramation: si en P le cône des tangentes est d'espèce s, en P_1 le cône des tangentes est d'espèce au moins égale à s. En particulier si P est biplanaire P_1 est uniplanaire. Enfin si P est uniplanaire, P_1 est au moins triple pour V_{r-2}.

Lorsque V_{r-1}^n admet un point double P, la droite OP appartenant tout entière à V_{r-1}^n, la projection P_1 de P est au moins double pour V_{r-2}. Si en P le cône des tangentes est d'espèce s, et si OP n'appartient pas à son sommet, le point P_1 est uniplanaire pour V_{r-2} et le cône des tangentes inflexionnelles en P_1 est formé d'un hyperplan et d'un cône d'espèce au moins égale à s. Si le cône des tangentes en P est d'espèce s et si OP appartient à son sommet, P_1 est au moins triple pour V_{r-2} et le cône des tangentes en P_1 est formé d'un hyperplan et d'un cône qui est en général d'espèce s. Si P est biplanaire, OP n'appartenant pas aux deux hyperplans tangents, P_1 est un tacnode de V_{r-2}. Si P est biplanaire et si OP appartient aux deux hyperplans tangents, P_1 est triple triplanaire. Enfin si P est uniplanaire, P_1 est triple et le cône des tangentes est décomposé en deux hyperplans dont l'un est double. L'auteur examine ensuite des singularités plus complexes. *L. Gauthier* (Nancy).

Vaccaro, G. Le ipersuperficie d'ordine n con un punto $(n-2)$-plo. II. Singolarità della ipersuperficie dedotte da quelle della varietà di diramazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 314–321 (1947).

Cette note a pour but d'établir les réciproques des propriétés étudiées dans la note analysée ci-dessus. Après une étude de la variété V_{r-4} de V_{r-3}^{2n-2} lieu des points doubles tracés des droites de V_{r-1}^n issues de son point $(n-2)$-uple O, l'auteur montre que toutes les singularités de la variété de diramation V_{r-2}^{2n-2} étudiées dans la note antérieure sont caractéristiques des singularités correspondantes de l'hyperplan double V_{r-1}^n. *L. Gauthier* (Nancy).

Gauthier, Luc. Sur certains systèmes linéaires de droites hyperspatiaux. Mém. Soc. Roy. Sci. Liége (4) 6, 371–552 (1945).

Le mémoire s'occupe de la détermination des congruences linéaires de droites dans un espace à r dimensions, c'est-à-

dire des systèmes ∞^{r-1} de droites de S_r tels qu'il passe une seule droite du système par un point générique de l'espace. On connaît déjà la solution complète de ce problème pour $r=2$, 3, 4. Dans le chapitre I l'auteur répète, sous forme soit géométrique soit analytique, les propriétés fondamentales des espaces tangents à une variété réglée le long d'une de ses génératrices rectilignes et celles des variétés focales, soit propres soit singulières, d'une congruence de droites. Dans le chapitre II ces propriétés générales trouvent leur application au cas des congruences linéaires; il s'agit alors de trouver dans S_r des variétés algébriques V_{r-2} dont les droites $(r-1)$-sécantes forment une congruence linéaire. On parvient à trois relations numériques fondamentales entre les caractères projectifs de la congruence et de ses variétés focales propre et singulières. Dans cette étude ce qui joue le rôle principal est une transformation birationnelle, déjà considérée par Ascione [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 6_1, 162–169 (1897)], que l'on obtient entre deux hyperplans génériques en considérant comme points homologues leurs intersections avec une même droite de la congruence. Le chapitre III est consacré au cas où la variété focale propre se compose de ∞^1 espaces linéaires; ce cas a une signification particulière car il comprend (si $r>4$) tous les cas possibles d'une variété focale à sections curvilignes rationnelles. Le chapitre IV passe méthodiquement en revue les cas d'une variété focale propre à sections curvilignes elliptiques; on trouve ainsi différentes solutions dans un espace à 5 dimensions. Dans le dernier chapitre on trouve enfin plusieurs constructions qui conduisent à des solutions générales valables dans un espace à un nombre quelconque de dimensions. Toutes les solutions considérées conduisent à des variétés focales rationnelles et qui constituent une partie de la base d'un faisceau d'hypersurfaces d'ordre $r-1$. *E. G. Togliatti* (Gênes).

Longo, C. Sui sistemi di ipersuperficie di S_r aventi lo stesso sistema primo polare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 282–287 (1947).

L'étude des systèmes d'hypersurfaces V_{r-1}^n qui ont le même système de premières polaires a déjà fait l'objet de recherches de Bertini [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (5) 7_2, 217–227, 275–281 (1898)]. L'auteur apporte ici un complément à cette étude. Après avoir rétabli quelques propriétés générales du système linéaire Λ formé par les V_{r-1}^n qui ont même système de premières polaires, et des espaces unis de l'homographie Ω qui associe les pôles d'une même polaire par rapport à deux V_{r-1}^n, il étudie complètement les systèmes Λ pour $r=3$ dans le cas où Ω a une seule racine caractéristique. *L. Gauthier* (Nancy).

Jongmans, François. Sur les complexes linéaires de quartiques gauches rationnelles dans l'espace à quatre dimensions. Mém. Soc. Roy. Sci. Liége (4) **6**, 1–56 (1945).

On trouve ici une généralisation des recherches de Stuyvaert sur les congruences linéaires de cubiques gauches dans l'espace ordinaire à trois dimensions [Mém. Soc. Roy. Sci. Liége (3) **7** (1907)]. L'auteur considère, plus en général, la variété rationnelle V_{r-3}^4 dans l'espace S_r à r dimensions qui est représentée par les équations:

$$(1) \qquad \begin{Vmatrix} a_z & b_z & c_z & d_z \\ a_{z'} & b_{z'} & c_{z'} & d_{z'} \end{Vmatrix} = 0,$$

où $a_z, b_z, \cdots, d_{z'}$ sont des formes linéaires en les coordonnées projectives x_i du point x. En supposant que les coefficients de ces formes renferment linéairement quatre paramètres $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, de sorte qu'en donnant aux x_i des valeurs

générales les équations (1) soient vérifiées par un seul système de valeurs des λ_i, on obtient une congruence linéaire de V_{r-3}. L'auteur détermine ensuite tous les types différents d'une telle congruence dans le cas $r=4$; il trouve en total 17 types; dans plusieurs de ces cas il est possible de transformer la congruence donnée de courbes en une congruence linéaire de droites à l'aide d'une transformation birationnelle (4, 4). Enfin, des simples modifications permettent de donner aux résultats obtenus, une signification générale pour un r quelconque. *E. G. Togliatti* (Gênes).

Jongmans, F. Contribution à la théorie des variétés algébriques. Mém. Soc. Roy. Sci. Liége (4) **7**, 367–468 (1947).

Ce travail s'occupe surtout de certaines limitations entre les caractères d'une surface ou d'une variété algébrique; quelques-unes d'elles étaient déjà connues et sont améliorées; d'autres sont des généralisations nouvelles de relations connues relatives aux courbes algébriques. Dans le chapitre I, en modifiant un peu un raisonnement de B. Segre, l'auteur obtient une borne supérieure nouvelle pour le nombre des moduls d'une surface algébrique. Le chapitre II est destiné à la recherche d'une borne inférieure pour le genre arithmétique p_a et d'une borne supérieure pour le genre géométrique p_g d'une surface algébrique; ce qui donne aussi une borne supérieure pour l'irrégularité de la surface. Toutes ces bornes sont exprimées en fonctions des caractères d'un système linéaire de courbes donné sur la surface (genre, degré, dimension); le point de départ est à chercher dans un travail classique de G. Castelnuovo sur les multiples minima d'une série linéaire donnée sur une courbe algébrique. Les bornes obtenues ne sont pas atteintes, mais elles sont très satisfaisantes. Le chapitre III s'occupe de la borne inférieure du genre linéaire d'une surface algébrique; l'auteur donne des améliorations des formules bien connues de Noether et de Castelnuovo, surtout dans le cas des surfaces qui possèdent des sous-multiples du système canonique. On en déduit comme application un nouveau système d'inégalités entre les caractères d'une variété algébrique à trois dimensions. Enfin, le chapitre IV s'occupe de savoir si les hypersurfaces d'un tel ordre découpent un système complet sur une variété algébrique donnée, ou s'il passe des hypersurfaces d'un tel ordre par la même variété; on parvient, entre autres, comme application, au théorème suivant. Une variété algébrique V_d, sans variété multiple V_{d-1}, dont la courbe-section générique est intersection complète de formes de son espace, est elle-même intersection complète de formes. *E. G. Togliatti* (Gênes).

Jongmans, F. Observations sur les courbes algébriques hyperspatiales. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 548–555 (1947).

Le but de cette note est de détailler complètement une démonstration contenue dans une recherche antérieure [voir l'analyse ci-dessus]. On démontre ici que sur une courbe irrationnelle d'ordre $n < 8$, sans singularités, appartenant à un espace quelconque S_r, les hypersurfaces d'ordre $n-r$ découpent une série complète non spéciale. La démonstration comporte trois cas différents selon que $r=3$, $r=4$, $r>4$. *E. G. Togliatti* (Gênes).

Lage Sundet, Knut. Some inequalities for plane Cremona transformations. Norsk Mat. Tidsskr. **30**, 17–21 (1948). (Norwegian)

If a linear system S of ∞^2 plane curves of degree n determines a Cremona transformation with the fundamental

points $P_1, \cdots, P_i$, then we have the relations

$$(1) \qquad \sum_{i=1}^{i} r_i^2 = n^2 - 1, \quad \sum_{i=1}^{i} r_i = 3(n-1), \quad r_1 + r_2 + r_3 > n,$$

where r_i is the order of singularity for S at P_i. A quadratic Cremona transformation having the fundamental points P_1, P_2, P_3 maps S into a system S'. By writing down the relations (1) for S' the author finds some new inequalities for n and the r_i which enable him to prove the following results. (I) There are no Cremona transformations with $r_1 > 1$, $r_2 > 1$ and $r_3 = r_4 = \cdots = r_i = 1$. (II) The only Cremona transformations with $r_1 > 1$, $r_2 > 1$, $r_3 > 1$ and $r_4 = \cdots = r_i = 1$ are those for which $n = 4$, $i = 6$ and $r_1 = r_2 = r_3 = 2$. (III) The only Cremona transformations with $r_1 > 1$, $r_2 > 1$, $r_3 > 1$, $r_4 > 1$ and $r_5 = \cdots = r_i = 1$ are those for which $i = n + 2$ and either $r_1 = r_2 = r_3 = \frac{1}{2}n$, $r_4 = \frac{1}{2}n - 1$ or $r_1 = \frac{1}{2}(n+1)$, $r_2 = r_3 = r_4 = \frac{1}{2}(n-1)$.   *F. J. Terpstra.*

Metelka, Josef. **Sur quelques groupes finis, composés des transformations de Cremona du 1ᵉʳ et du 5�knₑ ordre.** Acta Fac. Nat. Univ. Carol., Prague no. 174, 11–16 (1947). (Czech and French)

This paper summarizes the results of an investigation into the following problem: to find every possible case of a set of six points in the plane which are the principal points of two or more Cremona involutions of order 5 and of the first species, and to find the finite group of Cremona transformations arising from each such case. [The corresponding problem for involutions of the second species was treated by Bydžovský.] Six cases in all are found; and the associated groups, of orders 4, 6, 12, 24, 36, 120 respectively, are composed of the relevant involutions of order 5 and of various sets of cyclic collineations. In each case the invariant curves and the structure of the group are investigated.
J. G. Semple (London).

Differential Geometry

✱**Haack, Wolfgang.** **Differential-Geometrie. Teil I.** Wolfenbütteler Verlagsanstalt G.m.b.H., Wolfenbüttel and Hannover, 1948. 136 pp.

Familiar topics in the Euclidean geometry of curves and surfaces are studied, assuming a background knowledge of analytic geometry, differential and integral calculus. The relationship between differential geometry and the theory of the invariants of differential forms guides the development of the theory. The style is concise and clear, and the explanations and derivations are almost always motivated by intuitive geometric considerations. A simple vector notation is used, the discussion of covariant differentiation methods being reserved for volume II. There is an index at the end, but there are no exercises for the reader. The following extracts from the table of contents will indicate the scope of the book. (A) Vektoren und Bewegungsinvarianten. (B) Kurven im Raume. (C) Elemente der Flächentheorie. (D) Abbildungen zweier Flächen aufeinander (Flächentreue Abbildungen; Kartenprojektion; Konforme Abbildungen). (E) Ableitungsgleichungen und Integrierbarkeitsbedingungen (Ableitungsgleichungen von Gauss; Ableitungsgleichungen von Weingarten; Die Integrabilitätsbedingungen; Der Fundamentalsatz der Flächentheorie). (F) Geometrie auf der Fläche (Geodätische Linien, geodätische Krümmung; Parallelismus von Levi-Civita; Anwendung der Parallel-verschiebung von Levi-Civita zur Deutung der geodätischen Krümmung; Integralformel von Bonnet; Flächen konstanten Krümmungsmasses). (G) Minimalflächen.
A. Schwarz (New York, N. Y.).

Buzano, Piero. **La geometria differenziale in Italia (dal 1939 al 1945).** Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 12, 27 pp. (1946).

Hostinský, Bohuslav. **On the curvature of surfaces.** Rozpravy II. Třídy České Akad. 51, no. 4, 14 pp. (1941). (Czech)

The area P' of the spherical representation of the convex spherical polygon with the sides $a_1, \cdots, a_n$ is given by $P' = 2\pi - \sum_{i=1}^{n} a_i$ [theorem of Descartes]. The author extends the validity of this theorem to closed two-sided belts formed from n cyclically ordered planes and also to closed belts on developable surfaces. Next the author defines the curvature of an n-corner (it is the spatial angle of the polar n-corner associated with the given n-corner) and proves the following theorem. Let there be given the point $M(u, v)$ of an analytic surface with the curvilinear coordinates u, v and with the curvature K at the point M; let T be the total curvature of the four-cornered pyramidal surface with the vertex M and the sides AM, BM, CM, DM, where the points $ABCD$ of the surface have the coordinates $A(u+h, v)$, $B(u, v+k)$, $C(u-h, v)$, $D(u, v-k)$; let δ be the limit as $h, k \to 0$ of the ratio of the distance of the point $N(u+h, v+k)$ of the surface from the plane MAB to the area σ of the element $MANB$ of the surface. Then $\lim_{h \to 0, k \to 0} (T/\sigma) - K = \delta^2$. The author's proofs are often intuitive and not completely rigorous.
F. Vyčichlo (Prague).

Wunderlich, Walter. **Über die Böschungslinien auf Flächen 2. Ordnung.** Akad. Wiss. Wien, S.-B. IIa. 155, 309–331 (1947).

A helix is a skew curve whose tangents form a constant angle with a given direction. Thus the tangents intersect the improper plane in a circle c. The author bases his discussion of the helices on a quadric Φ on the following observation. Suppose Φ is nondegenerate and has the (proper or improper) center O. Let Π be an arbitrary plane, Π not containing O. The tangents of Φ through O intersect Π in a certain conic Φ_0, and the cone (or cylinder) polar to c with respect to Φ intersects Π in a second conic c^*. Then the helices belonging to c are projected from O into the evolvents of c^* with respect to the metric whose absolute conic is Φ_0.

"Die Betrachtungen sind in der Hauptsache rein geometrisch, geben jedoch die Mittel in die Hand, auch das mathematisch gefasste Problem einer Lösung zuzuführen, wie an Beispielen gezeigt wird. Dass die Darstellung gelegentlich etwas flüchtig ist, möge mit dem Hinweis auf die Vielfalt der auftretenden Fälle entschuldigt werden, deren eingehende Untersuchung sehr umfangreich würde. Sind doch—selbst bei Beschränkung auf reelle Lösungen—vom reell-affinen Standpunkt aus nicht weniger als 75 Typen zu unterscheiden, von denen manche noch eine Unterteilung erfahren könnten."   *P. Scherk* (Saskatoon, Sask.).

Sibirani, Filippo. **Superficie che contengono tre sistemi ∞^1 di eliche cilindrico-circolari ad assi paralleli.** Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (9) 10, 193–198 (1943).

By vector methods, the author studies the surface S defined by the parametric equations $x = a \cos u + b \cos v$,

$y=a \sin u+b \sin v$, $z=pu+qv$, where a, b, p, q are real constants such that $a>b>0$ and $p+q\neq0$. The curvilinear coordinates (u, v) determine a point on S with Cartesian coordinates (x, y, z). The parametric curves $u=u_1$ and $v=v_1$ form two families E_1 and E_2, each of ∞^1 circular helices. The axes of the helices of E_1 and E_2 are parallel. This surface S also contains another family E_3 of coaxial circular helices. Various properties of this surface S are discussed. For example, the Gaussian curvature is constant along each helix of the system E_3. Also the lines of curvature and asymptotic lines are determined in finite form. In the final part of the paper, the case where $p+q=0$ is discussed in detail. *J. De Cicco* (Chicago, Ill.).

Maneng, Louis. **Sur l'élément linéaire des surfaces de Weingarten.** C. R. Acad. Sci. Paris **226**, 1582–1584 (1948).

It is shown that the line element for a surface of Weingarten (except for one of constant mean curvature) can be written in the form $ds^2=(e^{-2\lambda}du^2+e^{2\lambda}dv^2)/f'(\lambda)$. The relation between this result and the classical form is developed and the line element of the spherical image of such surfaces is obtained. *C. B. Allendoerfer* (Princeton, N. J.).

Myller, A. **Indicatrice de troisième ordre de la courbure des surfaces.** Acad. Roum. Bull. Sect. Sci. **26**, 147–150 (1946).

The author considers the cylinders circumscribed about a surface S of ordinary space such that the generators through a given point P of S are tangent to S and make an angle ϕ with the direction of the principal normal curvature $1/R_1$. The projection of the curve of contact of any such cylinder and the surface S upon the tangent plane of S at P is termed a cylindrical curve of S at P. The cylindrical tangential curvature $1/\rho$ of S at P is the curvature of any cylindrical curve of S at P. This is a homogenous cubic polynomial in $\cos\phi$ and $\sin\phi$ with coefficients depending on third order magnitudes of S at P. By varying ϕ, the locus of the centers of curvature of the cylindrical curves of S at P is a cubic curve with an isolated singularity at P.

The problem is an extension of the following one considered by Mannheim. The normal cylindrical curvature $1/\lambda$ is the curvature of the section on the circumscribed cylinder perpendicular to the generators. Mannheim obtained the formula $\lambda=R_1\sin^2\phi+R_2\cos^2\phi$, where $1/R_1$ and $1/R_2$ are the principal normal curvatures. D'Ocagne observed that the relation between the normal cylindrical curvature $1/\lambda$ and the normal curvature $1/R$ is $\lambda R=R_1R_2$. *J. De Cicco* (Chicago, Ill.).

Ciobanu, Gh. **Sur une certaine surface réglée liée à une surface périsphère.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 117–120 (1947).

Let C_1 and C_2 be two curves in Euclidean 3-space and let C be a variable circle with center on C_1 and lying in the normal plane to C_1. The radius R of C is a function of the arc length s determining the position of its center on C_1. If P_1 denotes the general point of intersection of C_2 and a normal plane to C_1, consider the polar line D of P_1 with respect to the circle C in this plane. The lines D form a ruled surface. The author in this paper develops the conditions under which this ruled surface is a developable. The paper is marred by frequent typographical errors. *S. B. Jackson* (College Park, Md.).

Popa, Ilie. **Sur certaines surfaces de coïncidence.** Acad. Roum. Bull. Sect. Sci. **27**, 253–257 (1947).

Wilczynski [Trans. Amer. Math. Soc. **14**, 421–443 (1913)] showed that only on a particular coïncidence surface are the asymptotic curves of each family projectively equivalent. The object of this paper is to prove that this property is characteristic of this type of coïncidence surface. *V. G. Grove* (East Lansing, Mich.).

Krames, Josef. **Die Regelflächen dritten Grades mit einem Drehkegel als Zentraltorse.** Akad. Wiss. Wien, S.-B. IIa. **155**, 83–96 (1947).

In a previous paper [Akad. Wiss. Wien, S.-B. IIa. **133**, 65–90 (1924)] the author studied a class of ruled surfaces of the third order characterized by the following two properties: (i) the surfaces osculate the imaginary spherical circle in two points, (ii) they have as edge of striction (Striktionslinie) either a twisted cubic or a circle. In the present paper the author shows that the central developable of such a surface is a right circular cone. These surfaces are the only ruled surfaces of third order having as central developable a surface of constant slope (Böschungsfläche). The central developable of a ruled surface is the developable tangent to the ruled surface along its edge of striction. *E. Lukacs* (China Lake, Calif.).

Krames, Josef. **Über Regelflächen, die mit gewissen aus ihnen abgeleiteten Flächen kongruent sind.** Akad. Wiss. Wien, S.-B. IIa. **155**, 149–165 (1947).

Continuing his studies [see the preceding review] the author introduces the concept of related ruled surfaces. Two ruled surfaces are said to be related if they have the same edge of striction and the same central developable. It is shown that ruled cubic surfaces having a right circular cone as central developable and a circle of this cone as edge of striction are congruent to their related surfaces. A kinematic generation of these surfaces is given. The author then investigates the class of surfaces which are congruent to their related surfaces. It is shown that these surfaces have a helix as edge of striction and a developable screw as central developable. *E. Lukacs* (China Lake, Calif.).

Tigano, O. **Sulle superficie isoterme-asintotiche.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 298–300 (1947).

Let $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$ denote the parametric equations of the surface S for which the curvilinear coordinates (u, v) define the asymptotic lines of S. Then x, y, z are solutions of equations of the form $x_{uu}=ax_u+bx_v$, $x_{vv}=px_u+qx_v$, where (a, b, p, q) are functions of (u, v) satisfying a set of three compatibility conditions of which the simplest is $a_v=q_u$. The surface S is isothermal-asymptotic if

$$\frac{\partial^2}{\partial u\partial v}\log b/p=0.$$

The author presents a new characterization of the isothermal-asymptotic surfaces by means of the projectivity π, defined as the product of the null osculating systems to the asymptotic lines of a nonruled surface S at a general point P. This projectivity π has been studied by R. Calapso [same Rend. Cl. Sci. Fis. Mat. Nat. (6) **13**, 350–353 (1931)]. The new result is that a congruence of lines conjugate to a surface S (the developables of the congruence determine a conjugate system on S) is transformed by the projectivity π into a congruence conjugate to S if and only if S is isothermal-asymptotic. *J. De Cicco* (Chicago, Ill.).

Ruscior, Stefania. Sur une certaine correspondance par plans tangents parallèles entre deux surfaces cerclées. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] 2, 26–34 (1947).

L'auteur étudie un cas particulier du problème de la recherche des couples de surfaces cerclées (S) et (S_1) se correspondant ponctuellement avec correspondance des cercles et parallélisme des plans tangents. Supposant que, pour (S) comme pour (S_1), les cercles générateurs soient dans les plans normaux aux courbes γ, γ_1 décrites par leurs centres, elle obtient des couples (S), (S_1) pour lesquels γ et γ_1 sont transformées de Combescure. Les cercles correspondants C et C_1 dans (S) et (S_1) sont centrés en deux points correspondants de γ et γ_1, et leurs rayons r et r_1 sont définis, en fonction des rayons de courbure R et R_1 de γ et γ_1, par les formules $r = CR/(R-R_1)$, $r_1 = CR_1/(R-R_1)$, où C est une constante arbitraire. Les droites joignant deux points correspondants M et M_1 de (S) et (S_1) engendrent des cônes lorsque M et M_1 décrivent deux cercles homologues et la détermination de la courbe lieu du sommet de ces cônes montre que cette courbe est en correspondance de Combescure avec γ et γ_1 lorsque (S) et (S_1) sont des surfaces canaux.   *P. Vincensini* (Besançon).

Kasner, Edward, and **De Cicco, John.** Harmonic transformations and velocity systems. Univ. Nac. Tucumán. Revista A. 6, 187–193 (1947).

Let $u = x+iy$, $v = x-iy$, and consider the differential equation

$$(1) \qquad v'' = A(u,v) + B(u,v)v' + C(u,v)v'^2 + D(u,v)v'^3.$$

The integral curves of (1) include in particular the velocity system of any positional field of force. It is shown that the integral curves of (1) correspond under a harmonic transformation T to the straight lines in the (X, Y)-plane if and only if A, B, C, D satisfy $AC - A_v = 0$, $AD - B_v = 0$, $AD + C_u = 0$, $BD + D_u = 0$. A harmonic function effecting such a transformation necessarily is the product of a conformal transformation by an affinity; and any velocity family in the (x, y)-plane to which the theorem applies is the complete isogonal family of an isothermal family of curves.   *E. F. Beckenbach* (Los Angeles, Calif.).

Löbell, Frank. Betrachtungen über Flächenabbildungen. I. Das Vectorpolynom $\lambda^2\mathfrak{j}_1 - \lambda\mathfrak{j} + \mathfrak{j}_2$. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1945/46, 175–183 (1947).

A mapping of one surface on another determines, at corresponding points, an affine transformation or affinity between the two tangent planes. Let the surfaces have the vector equations $x = x(u, v)$ and $y = y(u, v)$, where corresponding points have the same parameter values. In a previous paper [same S.-B. Math. Nat. Abt. 1943, 217–237 (1944); these Rev. 8, 89] the author has discussed the relative vector invariants $\mathfrak{j}_1 = x_u \times x_v$, $\mathfrak{j}_2 = y_u \times y_v$, and $\mathfrak{j} = x_u \times y_v - x_v \times y_u$, and the relative scalar invariant $j = x_u y_v - x_v y_u$. Using these same invariants he here continues the discussion of the affine mapping of the planes, particularly with reference to the vector polynomial $\mathfrak{l} = \lambda^2\mathfrak{j}_1 - \lambda\mathfrak{j} + \mathfrak{j}_2$. This polynomial proves to have a bearing on the circumstances under which the affine transformation is a parallel projection, and also on the fixed directions, if any, of this transformation.   *S. B. Jackson* (College Park, Md.).

Löbell, Frank. Über einige Integralinvarianten, die bei Flächenabbildungen auftreten. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1944, 107–132 (1944).

In an earlier paper [same S.-B. Math. Nat. Abt. 1943, 217–237 (1944); these Rev. 8, 89] the author has introduced relative differential invariants of the first order of a mapping of one surface on another which he denotes by $\mathfrak{j}_1$, $\mathfrak{j}_2$, $\mathfrak{j}$, and j [cf. the preceding review]. In this paper from these relative differential invariants he constructs integral invariants of the mapping. Other than the ordinary surface vectors and surface area integrals, these invariants are $\mathfrak{E} = \iint_\omega j\,du\,dv$, $E = \iint_\omega \mathfrak{c} \cdot \mathfrak{j}\,du\,dv$, $G = \iint_\omega j\,du\,dv$, and are called respectively Spreizvector, Spreizwert and Schiefe. These invariants are discussed and their geometric significance developed, especially as regards the induced affinity between tangent planes at corresponding points of the mapping.   *S. B. Jackson* (College Park, Md.).

Bompiani, Enrico. Sur les directions inflexionelles d'une transformation de De-Jonquières. Acad. Roum. Bull. Sect. Sci. 26, 1–3 (1946).

On sait que si O, O' sont deux points homologues dans une transformation ponctuelle entre deux plans π, π', il existe trois droites a, b, c passant par O et trois droites correspondantes a', b', c' passant par O' telles que à chaque courbe ayant en O un point d'inflexion et ayant pour tangente d'inflexion une des droites a, b, c correspond une courbe ayant en O' un point d'inflexion et ayant pour tangente d'inflexion la droite a', ou b', ou c' correspondante. Ce sont les directions principales de la transformation pour le couple O, O' de points correspondants. L'auteur donne ici une simple construction géométrique des directions principales pour une transformation de De Jonquières d'ordre n entre π, π'. Si O_2 est le point base multiple d'ordre $n-1$ et A_1, $\cdots$, $A_{2(n-1)}$ sont les points bases simples dans le plan π, une des directions principales partant de O est OO_2; les deux autres sont données par les tangentes en O de la courbe C' qui a en O un point double, qui passe par A_1, $\cdots$, $A_{2(n-1)}$ et qui a en O_2 les mêmes tangentes de la courbe d'ordre $n-1$ passant par A_1, $\cdots$, $A_{2(n-1)}$ et ayant en O_2 la multiplicité $n-2$. La construction dépend seulement des points bases de la transformation.   *E. G. Togliatti* (Gênes).

Villa, Mario. Direzioni d'osculazione e d'iperosculazione di due trasformazioni puntuali. Boll. Un. Mat. Ital. (3) 2, 188–195 (1947).

The following theorem is proved on two point-transformations between two linear spaces of the same dimensionality r: if the points O and $\bar{O}$ are two regular corresponding points of the two spaces in both transformations, and if these act in the same way on the neighborhoods of order s of O and $\bar{O}$, there are $\{(s+1)^r - 1\}/s$ corresponding directions such that the curvilinear differential elements of order $s+1$ tangent to those directions are transformed in the same way by two transformations. This is an extension of a result given by E. Bompiani [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13, 837–848 (1942); these Rev. 8, 219] for $r = 2$, $s = 2$.   *E. Bompiani* (Rome).

Degoli, Lando. Sulle trasformazioni puntuali fra due spazi ordinari. Boll. Un. Mat. Ital. (3) 2, 217–221 (1947).

Some elementary calculations relating to the neighborhoods of the second order of two regular corresponding points in a point-transformation between two linear spaces.   *E. Bompiani* (Rome).

Pírko, Zdeněk. The fundamental equation of the motion of a variable plane figure and its application in the theory of plane curves. Časopis Pěst. Mat. Fys. **72**, D83–D86 (1947). (Czech)

Colmez, Jean. Recherches récentes sur les systèmes triples orthogonaux. Revue Sci. **85**, 1061–1062 (1947). Exposition of recent results of Bouligand and Llensa.

Câmpan, Florica. Sur l'indicatrice sphérique de Gauss et la relation entre les trois formes quadratiques d'une surface. Revista Științifică "V. Adamachi" **33**, 149–150 (1947).

Segre, B. Gruppi misti, ed orientazioni in geometria proiettiva differenziale. Ann. Mat. Pura Appl. (4) **25**, 313–324 (1946).

If in a geometry according to F. Klein G is the fundamental group, the stability group of a figure F is the group of all transformations of G leaving F invariant. Now Cartan [Bull. Soc. Math. France **69**, 47–70 (1941); these Rev. **7**, 164] has drawn attention to the fact that F can only be oriented if its stability group is mixed and that there exist just S orientations if this group can be decomposed into just S continuous sets of transformations. In the first part of this paper the theory of mixed groups is developed and all mixed groups are determined which are contained in the projective group on a straight line. Then the result of Cartan that a general curve element of order 5 in a complex projective plane has three orientations is duly reestablished and by means of three invariant cones the author solves the problem put by Cartan, to find a rather simple geometric interpretation of these orientations. This result leads to a new geometric interpretation of the projective curve element and the projective curvature of a plane curve.
J. A. Schouten (Epe).

Bell, P. O. Power series developments for the equations of a general analytic variety in hyperspace. Duke Math. J. **15**, 207–218 (1948).

Considérons une variété V_m d'un espace projectif S_n et associons à chaque point x_0 de cette variété un repère R projectif défini par $n+1$ points $x_0, \cdots, x_n$ et tel que les points $x_0, \cdots, x_m$ soient dans une variété plane à m dimensions tangente à V_m. Soient (1) $z^r = c'_{\alpha\beta}z^\alpha z^\beta + c'_{\alpha\beta\gamma}z^\alpha z^\beta z^\gamma + \cdots$ $(r = m+1, \cdots, n; \alpha, \beta, \gamma, \cdots = 1, \cdots, m)$ les équations en coordonnées non homogènes de V_m dans le repère R. Les coordonnées $z^1, \cdots, z^n$ d'un point quelconque Q dépendent de x_0, soit des paramètres $u^1, \cdots, u^m$ de x_0 sur V_m. Si Q est indépendant de x_0, on a (2) $\partial z^i / \partial u^\alpha = z^i z^i \Gamma^0_{i\alpha} - z^i \Gamma^i_{i\alpha}$ où les Γ sont définis par les relations $\partial x_i / \partial u^\alpha = \Gamma^h_{i\alpha} x_h$ $(i, h = 0, 1, \cdots, n)$.

En dérivant (1) par rapport à u^α (en tenant compte que les $c'_{\alpha\beta}, \cdots$ dépendent aussi de x_0) et remplaçant les dérivées des z par leurs expressions tirées de (2) on obtient après identification des relations de récurrence qui déterminent les coefficients $c'_{\alpha\beta}, \cdots$ de (1).

Le résultat est appliqué aux courbes gauches d'un S_3 dont le développement en série est calculé jusqu'au terme du neuvième degré et aux surfaces d'un S_3 dont le développement est calculé jusqu'au terme du sixième degré.
M. Haimovici (Jassy).

Arghiriade, E. Sur un théorème de Halphen. Acad. Roum. Bull. Sect. Sci. **28**, 83–87 (1945).

Halphen proved [Bull. Soc. Math. France **3**, 28–37 (1874)] that if a surface S has, at one of its points P, third order contact with a quadric surface, then P is a double point of the flecnodal curve on S. The object of this paper is to prove a generalized theorem of Halphen, namely, that if the surface has contact at P of the nth order with a quadric then P is a multiple point of the flecnodal curve on S of order $2(n-2)$. The converse theorem is not true [see Arghiriade, Bull. Math. Soc. Roumaine Sci. **45**, 55–62 (1943); these Rev. **7**, 31].
V. G. Grove.

Vincensini, Paul. Sur certains cônes quadratiques issus des points d'une hypersurface de l'espace euclidien à n dimensions. C. R. Acad. Sci. Paris **226**, 1069–1071 (1948).

Let S be a hypersurface in Euclidean space of n dimensions and to each point M let there be associated an orthogonal n-uple $R(e_1, \cdots, e_n)$, e_n being normal to S. Suppose that S is deformable and $\bar{S}$ is one of its deforms and let $\bar{M}$ and $\bar{R}$ be the homologues of M and R. Associate arbitrarily but invariably a point P in the tangent hyperplane to S at M. It is found that the directions of displacements of M on S orthogonal to the corresponding displacements of P generate in a certain manner a cone of $n-2$ dimensions in the tangent hyperplane to S at M. These cones in general vary with the tangent hyperplanes of S, but they are invariant under deformations of S if each point P is in the corresponding tangent hyperplane. This is a generalization of a result concerning surfaces in a Euclidean space of three dimensions [same C. R. **224**, 520–522 (1947); these Rev. **8**, 487]. How such ideas may be used to study surfaces corresponding with orthogonality of elements, and other topics, is pointed out.
V. G. Grove (East Lansing, Mich.).

Vincensini, Paul. Sur une transformation des champs de vecteurs unitaires. C. R. Acad. Sci. Paris **226**, 1163–1165 (1948).

Let v^h be a unit vector field in a Euclidean space; $\xi^i(x)$ represents the transformation $x^i \rightarrow X^i = x^i + \xi^i(x)$. A closed curve γ determines together with the field v^h along this curve a ruled surface for which $\int v_i dx^i$ has a certain value. The transformation $(x, v) \rightarrow (X, v)$ will leave this expression invariant if $(\partial_{[i}\xi^i)(\partial_{j]}v_i) = 0$. A geometrical solution of this equation is given. Furthermore the author treats the analogous problem for line congruences.
J. Haantjes.

Vyčichlo, F. Contribution to a generalization of Beltrami's theorem. Rozpravy II. Třídy České Akad. **50**, no. 2, 9 pp. (1940). (Czech)

All curves on a surface σ having at a point P the same tangential vector v have a spatial vector u in common at P [Hlavatý, Differentialgeometrie, Noordhoff, Groningen-Batavia, 1939, p. 415; these Rev. **1**, 27]. If the tangential vector v varies at P the geometrical locus of u is an algebraic cone of the tenth degree in general. Let ^ik $(i = 1, 2)$ be the first curvature of the ith asymptotic line through P and let $^i\kappa$ be the first curvature of the ith branch of the intersection curve of σ and its tangential plane at P. If P is a hyperbolic point then u gives us the well-known theorem of Beltrami on ^ik and $^i\kappa$. If on the contrary P is a parabolic point, then u leads to $^ik = ^i\kappa$ at P and moreover one gets $^ik + ^i\kappa$, $^ik^i\kappa$ $(i, j = 1, 2, i \neq j)$ expressed at P in terms of v and σ. [These statements are the author's results except for some slight corrections.]
V. Hlavatý.

Creangă, I. Sur une famille de transformations entre deux surfaces réglées. Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] 2, 35–45 (1947).

Étant données deux surfaces réglées Σ_1, Σ_2, il existe une famille K de transformations ponctuelles de Σ_1 en Σ_2, dépendant de quatre fonctions arbitraires, conservant les génératrices et induisant une homographie entre chaque couple de génératrices correspondantes. Toute transformation T de K transforme une famille quelconque de courbes divisant homographiquement les différentes génératrices de Σ_1 (familles R) en une famille analogue sur Σ_2, et K peut être décomposée en deux classes, l'une formée par les T qui conservent les mêmes couples de génératrices de Σ_1 et Σ_2 en changeant les homographies induites sur les génératrices, l'autre par les T qui conservent ces homographies en changeant les couples de génératrices correspondantes. L'auteur envisage les transformations de la première classe. Considérant les T conservant deux lignes asymptotiques et faisant usage d'un repère projectif local approprié, il étudie, pour l'une de ces transformations, la congruence Γ' des droites joignant deux points correspondants. Les droites joignant les points homologues sur deux génératrices correspondantes g_1 et g_2 de Σ_1 et Σ_2 forment une demi-quadrique S_1; les génératrices de la demi-quadrique complémentaire S_2 déterminent une deuxième congruence Γ'^* contenant Σ_1 et Σ_2, et toutes les surfaces réglées extraites de Γ'^* peuvent être mises en correspondance T soit avec Σ_1 et Σ_2 soit entre-elles au moyen des droites de Γ', les rôles de Γ' et Γ'^* pouvant être intervertis. Si, pour tout couple (g_1, g_2), $Q = (S_1 + S_2)$ reste tangente à Σ_1 le long de g_1, Σ_1 est nappe focale de Γ' et la deuxième nappe focale est engendrée par une certaine cubique. L'auteur recherche les conditions pour que cette deuxième nappe focale se réduise à Σ_2. Il traite le cas particulier où Σ_1 est une demi-quadrique Q_1, et plus spécialement celui où Σ_2 est aussi une demi-quadrique Q_2. Dans ce dernier cas Q reste tangente à Q_1 et Q_2 suivant deux génératrices g_1, g_2 correspondantes.   *P. Vincensini* (Besançon).

Creangă, I. Sur la courbure conique. Acad. Roum. Bull. Sect. Sci. 25, 116–129 (1943).

The author develops the theory of the osculating cone and conical curvature at a point P of a curve C on a surface S of ordinary space, which was originally introduced by A. Myller. The limit of the cone of revolution inscribed in the trihedral angle formed by the three tangent planes of S at the points P, P', P'' of C, as both P' and P'' approach P along C, is called the osculating cone at a point P of a curve C on the surface S. The author finds it convenient to define the conical curvature as the tangent of the angle formed by the axis of the osculating cone and its generators instead of the cotangent of this angle which was the original definition of Myller. The following results are due to Myller. The osculating cone and the conical curvature are invariant under any dilatation (under any transformation between parallel planes of surfaces). The conical curvature (as defined by Creangă) is equal to the ratio of the curvature and torsion of the edge of regression of the developable surface circumscribed about S along C.

The author obtains the equation of the osculating cone and the analytic expression for the conical curvature. The theorems of Myller are proved. The conical curvature is $d\phi/d\theta$, where $d\phi$ is the angle between two tangent lines to C at two nearby points P and P' and $d\theta$ is the angle between the two respective conjugate directions at P and P'. The cylindrical curves of a surface S are those of zero conical

curvature. At each regular point P of S there are three directions (the conical asymptotic directions) for which the geodesics in these directions have zero conical curvatures. There is an indicatrix of conical curvatures of the geodesics which pass through a regular point P of S. This indicatrix, which is analogous to that of Dupin, is a quartic curve with a triple point at P. There are three extremes for the conical curvatures of the geodesics through P. The lines of conical curvature are the curves on S corresponding to these extreme directions. Finally the author studies the surfaces for which the conical curvatures of the geodesics are constant.   *J. De Cicco* (Chicago, Ill.).

Creangă, Ioan. Sur la correspondance par rayons parallèles entre deux congruences de droites. Acad. Roum. Bull. Sect. Sci. 26, 151–154 (1946).

Deux congruences rectilignes K, $\bar{K}$ se correspondant avec parallélisme des rayons homologues, l'auteur envisage les différents couples de surfaces réglées homologues passant par deux rayons homologues quelconques. La correspondance par plans tangents parallèles entre les deux surfaces d'un même couple est une homothétie, que l'auteur a étudiée dans une note antérieure [C. R. Acad. Sci. Roum. 1, 287–290 (1937)]. La note actuelle prolonge la précédente. Il y est établi que le lieu des centres d'homothétie relatifs aux différents couples de surfaces réglées issues de deux rayons homologues fixes quelconques r, $\bar{r}$ de K, $\bar{K}$ est une conique, et que cette conique est déterminée par les quatre foyers de K, $\bar{K}$ portés par r, $\bar{r}$ et par le point caractéristique du plan $[r, \bar{r}]$ lorsque le couple r, $\bar{r}$ varie.   *P. Vincensini*.

Grove, V. G. Pairs of rectilinear complexes. Amer. J. Math. 70, 364–374 (1948).

Cet article est consacré à l'étude des correspondances biunivoques entre les génératrices de deux complexes linéaires donnés. Dès le début la distinction est faite entre les couples de complexes gauches (pour lesquels les génératrices homologues ne se coupent pas) et les couples de complexes concourants (dont les génératrices homologues se coupent). En ce qui concerne les couples gauches (Γ_1, Γ_2), l'auteur envisage les deux faisceaux de complexes linéaires tangents respectivement à Γ_1 et Γ_2 le long de deux génératrices homologues quelconques g_1 et g_2, et la détermination des complexes L_1 et L_2 du premier et du deuxième faisceau, contenant, le premier g_2 et le deuxième g_1, l'amène à considérer deux cas suivant que la congruence rectiligne commune à L_1 et L_2 a ses directrices distinctes ou confondues, et à distinguer ainsi deux types de couples de complexes gauches. Dans les deux cas la considération des polarités définies par L_1 et L_2 conduit à un procédé de détermination d'un tétraèdre de référence covariant au couple (Γ_1, Γ_2), variable suivant le type envisagé, et permet d'associer à (Γ_1, Γ_2) des figures diverses covariantes, notamment des quadriques, des complexes tétraèdraux, et certaines courbes ou surfaces. La détermination du tétraèdre de référence covariant et des figures covariantes associées est reprise ensuite, avec les modifications qui s'imposent, pour les couples (Γ_1, Γ_2) concourants.   *P. Vincensini* (Besançon).

Backes, Fernand. Sur une infinité de systèmes cycliques attachés en un point d'une cyclide générale. C. R. Acad. Sci. Paris 226, 1680–1681 (1948).

In pentaspherical coordinates, a cyclide is given by the equation $\sum x_i^2/a_i = 0$, $i = 1, \cdots, 5$, where the a_i are constants. A parametric representation of a cyclide is

$_i = [a_i(a_i - u)(a_i - v)/f'(a_i)]^{\frac{1}{2}}$, where $f(t) = \prod_{i=1}^{4}(t - a_i)$ [see J. L. Coolidge, A Treatise on the Circle and the Sphere, Oxford, 1916]. The sphere S_h defined by the equation $\sum X_i \alpha_i(a_i^{-1} + h) = 0$, where h is constant, is tangent to the cyclide at the point $P(x_i)$. These are spheres S_h of Ribaucour. Denote the second characteristic point of these spheres S_h by P_h. The circle Γ_h orthogonal to S_h at P and P_h is given by the equations

$$(u + h)\sum X_i \frac{x_i}{a_i - u} = (v + h)\sum X_i \frac{x_i}{a_i - v} = -h\sum X \alpha_i.$$

These circles Γ_h describe a cyclic system. Each of these circles is on a sphere orthogonal to the cyclide at the point x_i. These new spheres are defined by the equation

$$\sum X_i \alpha_i a_i/(a_1 - u)(a_1 - v) = 0.$$

The parameters u and v represent the lines of curvature of the envelope of these spheres. The five coordinates of these spheres satisfy a Laplace equation of the Euler-Poisson type.

The case $h = 0$ is discussed in detail. Finally the author considers a family of spheres whose five coordinates are $x_i(1 + \lambda/(a_1 - u) + \mu/(a_1 - v))$, λ, μ constants. These expressions satisfy a Laplace equation of the Euler-Poisson type if and only if $\lambda = \mu$. In this way there is obtained a simple infinity of spheres cutting the cyclide orthogonally at the point x_i in such a way that (u, v) are the parameters of the lines of curvature of the envelope. *J. De Cicco.*

Backes, F. **Sur une extension de la notion de système cyclique.** Acad. Roy. Belgique. Bull. Cl. Sci. (5) **34**, 52–64 (1948).

The purpose of this paper is to generalize a theorem of Ribaucour which states that, if the circles of a congruence of circles are normal to more than two surfaces, they are normal to an infinity of surfaces on which the lines of curvature correspond. The generalization consists in replacing the circle by a central conic depending on two parameters and having the following properties. (1) The conic has an infinity of points P which generate surfaces whose tangent planes contain the normal to the plane of the conic through a focus F. (2) The conic has an infinity of other points P' generating surfaces whose tangent planes contain the normal to the plane of the conic through the second focus F'. (3) It contains moreover an infinity of points Q generating surfaces whose tangent planes pass through the normal to the plane of the conic through its center O. The method used is that of the moving trihedron of reference in two parameters. It is found that, if a point O describes an arbitrary surface of Monge, and if at O in the plane π of the plane line of curvature, which is also a geodesic, a line OX is drawn making an angle with the tangent to this geodesic which is an arbitrary function of the arc s of that curve, then a conic with center at O in the plane π having OX as the line of its foci and whose parameter and eccentricity are arbitrary functions of s is the most general conic having the three properties above. Moreover, the points F, F', P, P' and Q all describe surfaces of Monge whose lines of curvature which are geodesics are also in the plane π. *V. G. Grove* (East Lansing, Mich.).

Lagrange, René. **Sur les congruences de cercles du plan.** Bull. Sci. Math. (2) **71**, 82–104 (1947).

The purpose of this paper is to study a two-parameter family of circles in the plane. An example of such a family is the family of osculating circles to the one-parameter family of curves defined by the differential equation $y' = f(x, y)$. A typical theorem may be stated as follows. If a one-parameter family of curves is invariant under an inversion with center at O, and E is one of the family, the osculating circles of E are also the osculating circles of the inverse of E with respect to O. Generalizations are made of the notion of focal points, etc., which arise in the theory of congruences of lines in space. *V. G. Grove.*

Haimovici, Adolf. **Sur une certaine déformation des congruences de courbure nulle.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 22–25 (1947).

Dans une note précédente [même Bull. **1**, 238–255 (1946); ces Rev. **8**, 488], l'auteur a étudié quelques cas particuliers du problème de la déformation des congruences de sphères. Il envisage ici le problème de la détermination des congruences de sphères (A) susceptibles d'une déformation (conservant l'angle de deux sphères infiniment voisines) laissant invariants les cercles suivant lesquels chaque sphère A coupe une sphère fixe Q. Il montre que les congruences (A) font partie de la classe des congruences de courbure nulle étudiées par P. Vincensini [J. Math. Pures Appl. (9) **16**, 315–328 (1937)], et établit que les congruences de sphères $(\bar{A})$ déformées de (A) dépendent de deux constantes arbitraires a et b, l'expression de $\bar{A}$ pouvant, par un choix convenable des deux paramètres u_1, u_2 fixant A, être mise sous la forme

$$\bar{A} = \tanh (u_1 + a)A + \frac{e^{\pm u_1 + b}}{\cosh (u_1 + a)}Q.$$

P. Vincensini (Besançon).

Haimovici, M. **Sur une nouvelle espèce de connexions affines dans les espaces des familles simplement transitives de transformations de variables.** Bull. Math. Soc. Roumaine Sci. **47**, 35–48 (1946).

If a simply transitive set of transformations of n variables x^h into n variables X^h is given, in the $2n$-dimensional space of the x^h, X^h three invariant affine connections can be defined as was proved by the author in two notes which will appear elsewhere. If the set is the group of transformations of the parameters of another group these connections are intimately related to the connections introduced in 1926 by Cartan and Schouten. In this paper three other invariant connections are defined and it is shown that they correspond to another connection in group space, defined by Cartan. *J. A. Schouten* (Epe).

Haimovici, Mendel. **Sur les espaces des familles simplement transitives de transformations de variables à courbure de III-e espèce nulle.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iaşi] **2**, 46–70 (1947).

A toute famille simplement transitive de transformations $(x^i) \to (X^i): X^i = F^i(x^j, a^j)$, $i, j = 1, 2, \cdots, n$, l'auteur associe un espace à $2n$ dimensions de coordonnées (x^i, X^i) qui peut être doué de six façons d'une structure d'espace ponctule à connexion affine (avec torsion). Il détermine ici, sous forme entièrement finie, les transformations pour lesquelles certaines de ces connexions sont à courbure nulle. *A. Lichnerowicz* (Strasbourg).

Cossu, A. **Sulle connessioni affini e sul vettore di Einstein.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 295–298 (1947).

L'auteur démontre la propriété suivante du rotationnel du vecteur d'Einstein $S_{\cdot \cdot}^{\cdot \cdot} = \phi_\cdot$ d'un espace à connexion

affine. Ce rotationnel (multiplié intérieurement par un bivecteur infinitésimal) est égal au rapport entre (a) la différence des valeurs finales d'un volume V à n dimensions après le transport le long d'un circuit infinitésimal équivalent au bivecteur donné avec la connexion de l'espace et avec la connexion affine conjuguée (ou la connexion affine symétrique associée) et (b) le volume même. En particulier si ϕ_ω est un gradient, le rapport précédent est nul; dans ce cas, la connexion donnée a la même courbure que la connexion dont les coefficients sont $P_{\rho\sigma}^\lambda = \Gamma_{\rho\sigma}^\lambda + (2/(n-1))\delta_\rho{}^\lambda\phi_\sigma$.

M. *Haimovici* (Jassy).

Cossu, A. Alcune osservazioni sulle varietà subordinate di una varietà a connessione affine asimmetrica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 303–311 (1947).

Désignons par ∇ la connexion affine d'un espace A_n, par ∇^* la connexion affine conjuguée à ∇ et par $\nabla^{(b)}$ la connexion affine symétrique associée à ∇. La note établit les résultats suivants. Les connexions affines induites dans une variété X_m plongée dans A_n par ∇^* et $\nabla^{(b)}$ sont respectivement conjuguée et symétrique associée à celle induite par ∇. Si les premiers tenseurs réduits de courbure eulérienne de ces connexions induites sont égaux, la torsion associée à un cycle infinitésimal situé sur X_m est la même dans la connexion ∇ et dans celle induite par ∇ sur X_m. Si la connexion ∇ est semi-symétrique, la connexion induite est en général semi-symétrique; elle est symétrique si X_m est en chaque point tangent à l'hyperplan défini par le vecteur d'Einstein de A_n. Des propriétés du même genre sont établies pour la connexion induite sur le système de directions planes pseudo-normales à X_m.

M. *Haimovici* (Jassy).

Levine, Jack. Invariant characterizations of two-dimensional affine and metric spaces. Duke Math. J. **15**, 69–77 (1948).

L'auteur construit des systèmes de tenseurs dont l'annulation exprime qu'un espace à connexion affine sans torsion à deux dimensions: (1) admet un champ de vecteurs parallèles, (2) admet une intégrale première linéaire de l'équation des géodésiques, (3) est un espace riemannien. Les résultats (1) et (2) sont appliqués aux espaces riemanniens V_2. Ce travail fait suite à un autre travail publié en collaboration avec T. Y. Thomas [Bull. Amer. Math. Soc. **40**, 721–728 (1934)] où l'existence des conditions tensorielles de cette sorte est établie.

M. *Haimovici*.

Urban, A. Differential equations of curves on a special V_{n-1} in V_n. Rozpravy II. Třídy České Akad. **57**, no. 9, 12 pp. (1948). (Czech)

Let $a_{\lambda\mu}$ $(b_{\lambda\mu})$ be the first (second) fundamental tensor of a V_{n-1} in V_n and $b_{\lambda\mu}=ca_{\lambda\mu}$ ($c=$constant). Moreover, let i^ν (I^ν) be the unit tangential vector of a curve on V_{n-1} which is thought of as a curve of V_n (V_{n-1}). Then we have two sets of Frenet formulae, one (with the curvatures $k_1, \cdots, k_{n-1}$ and normal unit vectors $i_2^\nu, \cdots, i_n^\nu$) for i^ν and the other (with the curvatures $K_1, \cdots, K_{n-2}$ and normal unit vectors $I_2^\nu, \cdots, I_{n-1}^\nu$ in V_{n-1}) for I^ν. These two sets are related by a set S of decomposition formulae which one gets by taking the derivatives of the well-known decomposition formula $k_1 i_2^\nu = K_1 I_2^\nu + c n^\nu$ (n^ν being the unit normal vector to V_{n-1}). The set S admits the elimination of $K_1, \cdots, K_{n-2}$ and the resulting formula is a differential equation (containing only $k_1, \cdots, k_{n-1}$ and their derivatives) of a curve on our special V_{n-1}. If $n=3$, we get in this way the classical differential equation for spherical curves.

V. *Hlavatý*.

Bochner, S. Curvature and Betti numbers. Ann. of Math. (2) **49**, 379–390 (1948).

The reviewer has proved [Duke Math. J. **8**, 401–404 (1941); these Rev. **3**, 18] that if the Ricci (mean) curvature of a compact Riemannian manifold M^n is everywhere positive, the fundamental group of M^n is finite. It follows that the first Betti number B_1 of M^n is 0 and this result has been proved again by the author [Bull. Amer. Math. Soc. **52**, 776–797 (1946); these Rev. **8**, 230] using the theorem of Hodge that the vanishing of B_1 is equivalent to the nonexistence of a harmonic vector field. In the present paper the author uses the further result of Hodge that the vanishing of B_p is equivalent to the nonexistence of a harmonic tensor field of order p; by hypothesizing conditions on various curvatures of a compact M^n, the nonexistence of harmonic tensors is proved, with consequent vanishing of Betti numbers. The main theorem states that if the Ricci curvature of M^n is positive, and if for some p $(1 \leq p \leq n-1)$ the relation

$$(p-1)|C| < (1-2(p-1)/(n-2))L + (p-1)|R|/\{(n-1)(n-2)\}$$

holds at all points P, then $B_p = 0$ for that p. Here $|C|$ is the supremum over all skew-symmetric tensors η^{ij} at P of the expression $\eta^{hk}\eta^{ij}C_{hijk}/(\eta_{hk}\eta^{hk})$, where C_{hijk} is the conformal curvature tensor, L is the smallest Ricci curvature at P and R is the scalar curvature at P. In spaces where the Ricci curvature is known to be constant positive and certain Betti numbers not zero, a positive lower bound for $\sup |C|$ over M^n is deduced from this result. Examples of such spaces are products of spheres, and compact semi-simple groups.

S. B. *Myers* (Ann Arbor, Mich.).

Lichnerowicz, André. Courbure et nombres de Betti d'une variété riemannienne compacte. C. R. Acad. Sci. Paris **226**, 1678–1680 (1948).

The author states some results complementary to results of Bochner [see the preceding review] and deducible by the same methods. For example, if a compact orientable Riemannian manifold V^n is isometrically imbeddable in a V^{n+1} of constant nonnegative curvature so that its second fundamental form is definite, then $B_1(V^n) = B_2(V^n) = 0$.

S. B. *Myers* (Ann Arbor, Mich.).

Blum, Richard. Ueber die Bedingungsgleichungen einer Riemann'schen Mannigfaltigkeit, die in einer Euklidischen Mannigfaltigkeit eingebettet ist. Bull. Math. Soc. Roumaine Sci. **47**, 144–201 (1946).

The author considers the problem of imbedding a Riemann space V_n in a Euclidean space. His paper constitutes a generalization of the work of G. Vranceanu [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **11**, 385–389 (1930)] and especially of that of T. Y. Thomas [Acta Math. **67**, 169–211 (1936)]. It is shown that, in general, (1) the Ricci equations are consequences of the Gauss-Codazzi equations; (2) if V_n is of class ν and $0 < \nu \leq n(n-2)/8$, then the Codazzi equations are consequences of the Gauss equations; (3) if $n(n-2)/8 < \nu \leq n(n-1)/2$, then $n^2(n^2-1)(n-2)/24$ Codazzi equations are consequences of the Gauss equations and of the remaining Codazzi equations. Necessary and sufficient conditions that a V_n be of class ν are derived, leading to two regular cases and one singular case. For Riemann spaces of class one, the type number τ, defined by T. Y. Thomas in the above reference, is introduced. In addition to Thomas's discussion for $\tau \geq 4$ and $\tau = 3$, further results for the cases $\tau = 3$, $\tau = 2$ are obtained.

A. *Fialkow* (Brooklyn, N. Y.).

Borș, Const. **Sur les courbes concurrentes dans la géométrie centro-affine.** Bull. École Polytech. Jassy [Bul. Politehn. Gh. Asachi. Iași] **2**, 71–78 (1947).

The background for this paper lies in a work of O. Mayer and A. Myller [Ann. Sci. Univ. Jassy **18**, 234–280 (1933)]. There distance of two points (x_1, y_1), (x_2, y_2) was defined as $x_1y_2 - x_2y_1$ with a correlative definition of angle. This led to a natural generalization of parallel curves (equidistant and with parallel tangents). The author extends this line of investigation to a generalization of "concurrent curves" (equidistant with tangents making a constant angle). The problem of finding all curves concurrent to a given one is reduced to the solution of a certain Riccati equation. Some centro-affine invariants of concurrent curves are exhibited. A number of situations in which the Riccati equation can be readily solved are given individual attention, for example, parallel curves, homothetic curves and curves of zero distance. *J. L. Vanderslice* (College Park, Md.).

Míkan, Milan. **Non-Euclidean line geometry.** Rozpravy II. Třídy České Akad. **55**, no. 6, 45 pp. (1945). (Czech)

Let K be the quadratic image in P_5 of the four-dimensional line space in P_3, where P_3 is a projective image of a non-Euclidean three-space with the absolute quadric N. The quadratic (special) complex of all tangents to N is represented in P_5 by a quartic surface W (on K) which may be thought of as a base of a one-parameter set S of four-dimensional quadrics in P_5. Hence the non-Euclidean line geometry of P_3 is equivalent to the theory of invariants of S in P_5. The author uses the reviewer's method [Differentielle Liniengeometrie, Noordhoff, Groningen, 1945; these Rev. **8**, 346] on two quadrics of the set S (one of which is K) in order to investigate the problems of non-Euclidean line geometry of P_3. The results cannot be described in detail owing to a set of rather complicated preliminary notions. *V. Hlavatý* (Bloomington, Ind.).

NUMERICAL AND GRAPHICAL METHODS

Kitover, K. A. **Tables of the sums of certain infinite trigonometric series.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 233–240 (1948). (Russian)

Tables are given of

$$F_{2r}(x) = \sum_{n=1}^{\infty} n^{-2r} \sin nx,$$

$$F_{2r+1}(x) = \sum_{n=1}^{\infty} n^{-2r-1} \cos nx,$$

$$f_{2r}(x) = \sum_{n=1}^{\infty} (2n-1)^{-2r} \sin (2n-1)x,$$

$$f_{2r+1}(x) = \sum_{n=1}^{\infty} (2n-1)^{-2r-1} \cos (2n-1)x.$$

Here F_k and f_k are connected by the relation

$$f_k(x) = F_k(x) - 2^{-k}F_k(2x)$$

and are tabulated for $k = 1(1)6$ and for $x = m\pi/180$, $m = 0(1)180$. Values are given to 4 decimals. A number of similar trigonometric sums are expressed in terms of $F_k(x)$. Except for $F_1(x) = -\log (2 \sin \frac{1}{2}x)$, $f_1(x) = \log \cot \frac{1}{2}x$, the functions F_k and f_k are higher transcendants. They are said to occur in elasticity theory. *D. H. Lehmer.*

Tietze, Heinrich. **Zusammenstellung einiger Werte des Integrallogarithmus.** S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. **1947**, 47–50 (1947).

This paper contains the values of the function Li (x) for $x = k \times 10^n$, $k = 1(1)9$, $n = 0(1)4$ and $x = 10^5$ to 8 significant figures. Two additional values Li $(200000) = 18036.1$, Li $(300000) = 26086.4$, due to Weigand, are also given. *D. H. Lehmer* (Berkeley, Calif.).

Ghizzetti, Aldo. **Sul calcolo di un integrale che compare nella teoria della produzione di coppie di elettroni.** Univ. Nac. Tucumán. Revista A. **6**, 37–50 (1947).

The integral in question (actually the sum of a simple integral and an iterated integral) is a function of a real parameter k which runs from 4 to ∞: the integrands are very involved combinations of elementary (algebraic and logarithmic) functions. In preparation for numerical computation, the author uses rather ingenious transformations in order to express his integral in terms of elliptic integrals and of two integrals in the integrands of which elliptic integrals occur. He gives approximate formulae for k near to 4 and k very large, and a short numerical table. *A. Erdélyi* (Edinburgh).

Ghizzetti, A. **Tavola della funzione euleriana $\Gamma(z)$ per valori complessi dell'argomento.** Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3**, 254–257 (1947).

This note contains a little table to 5 figures of $\Gamma(x+iy)$, x ranging from 4 to 5 at intervals of 0.1, y ranging from 0 to 1 at intervals of 0.1. For interpolation the marginal values $3.9+0.2ni$, $5.1+0.2ni$, $4.0-0.1i+0.2n$, $4.0+1.1i+0.2n$ $(n = 0, 1, 2, 3, 4, 5)$ are inserted. The values within the square (x_0, y_0), (x_0, y_1), (x_1, y_1), (x_1, y_0) $(x_1 = x_0 + 0.1, y_1 = y_0 + 0.1)$ may be obtained to 2 exact decimals by the bilinear interpolation formula

$$\begin{aligned}\Gamma(x+iy) = 100\{&(x_1-x)(y_1-y)\Gamma(x_0+iy_0) \\ &+ (x-x_0)(y_1-y)\Gamma(x_1+iy_0) + (x_1-x)(y-y_0)\Gamma(x_0+iy_1) \\ &+ (x-x_0)(y-y_0)\Gamma(x_1+iy_1)\}.\end{aligned}$$

To obtain more exact results (to 4 decimals) the 5 values z_0, $z_1 = z_0 + 0.1$, $z_2 = z_0 + 0.1i$, $z_3 = z_0 - 0.1$, $z_4 = z_0 - 0.1i$ are used in Lagrange's interpolation formula:

$$\Gamma(z) = 2500 \prod_{k=0}^{4}(z-z_k)\left\{ \sum_{k=1}^{4} \frac{\Gamma(z_k)}{z-z_k} - \frac{4\Gamma(z_0)}{z-z_0} \right\}.$$

The values of $\Gamma(z)$ outside the given square may be obtained by the familiar functional equations. *S. C. van Veen* (Delft).

Kaplan, E. L. **Auxiliary table for the incomplete elliptic integrals.** J. Math. Physics **27**, 11–36 (1948).

Interpolation into Legendre's tables for the incomplete elliptic integrals

$$F(k, \alpha) = \int_0^{\alpha} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}, \qquad E(k, \alpha) = \int_0^{\alpha} \sqrt{1 - k^2 \sin^2 \varphi} \, d\varphi$$

[A. M. Legendre, Exercices de Calcul Intégral, v. 3, Paris, 1816; Traité des Fonctions Elliptiques, v. 2, Paris, 1826] is not practicable when $\sin \alpha$ and k are near 1. The author derives and tabulates easily interpolable functions from

which the integrals may be calculated in this case, especially:

$$K - F(k, \alpha) = 2\pi^{-1}K' \sinh^{-1} x/r + x(r^2+x^2)^{\frac{1}{2}}f,$$

where

$$\frac{f}{(1+r^2)^{\frac{1}{2}}} = \sum_{m=0}^{\infty}\left(\frac{2\cdot4\cdots\cdot2m}{3\cdot5\cdots\cdot(2m+1)}\right)x^{2m}$$
$$\times\left\{\sum_{l=m}^{\infty}(-1)^{l-m}\left(\frac{1\cdot3\cdots\cdot(2l+1)}{2\cdot4\cdots\cdot(2l+2)}\right)^2 r^{2(l-m)}\right\},$$

and

$$E - E(k, \alpha) = 2\pi^{-1}(K' - E') \sinh^{-1} x/r + x(r^2+x^2)^{\frac{1}{2}}e,$$

where

$$e(1+r^2)^{\frac{1}{2}} = x^{-1}\sin^{-1}x + \frac{1}{2}\sum_{m=0}^{\infty}\left(\frac{2\cdot4\cdots\cdot2m}{3\cdot5\cdots\cdot(2m+1)}\right)x^{2m}$$
$$\times\left\{\sum_{l=m-1}^{\infty}(-1)^{l-m}\left(\frac{1\cdot3\cdots\cdot(2l+1)}{2\cdot4\cdots\cdot(2l+2)}\right)^2\frac{r^{2(l-m)}}{l+2}\right\},$$

$x = \cos\alpha$, $r = (k^{-2}-1)^{\frac{1}{2}}$. The principal table gives 10-decimal values of f and e for $x^2 = -.005(.005)0.16$, $r^2 = -.005(.005)-0.16$, the negative arguments being included to facilitate interpolation. Tabular errors should not exceed 1×10^{-10}. As a by-product of the principal computations, 12-decimal values of the functions $2K'/\pi$ and $2(K' - E')/\pi$ are obtained as functions of r^2 from $r^2 = -.005(.005)0.16$ and are included here. The author has carefully checked Legendre's tables I and IX. A table of errata is inserted.   *S. C. van Veen.*

✱Cambi, Enzo. **Eleven and Fifteen-Place Tables of Bessel Functions of the First Kind, to All Significant Orders.** Dover Publications, Inc., New York, N. Y., 1948. vi+154 pp. $3.95.

The main table gives $J_n(x)$, $n=0(1)29$, for x varying from 0 to 10.5 at an interval of 0.01 to 11 places. The last column (headed "checks") contains two lists of figures: the first gives, in units of the 11th place, the error encountered in checking the formula (1) $1 = J_0 + 2\sum_{k=1}^{\infty}J_{2k}$; the second, that encountered in checking the formula $\sin x = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}$. A supplementary table gives $J_n(x)$, $n=0(1)11$, for x varying from 0 to 0.5, at an interval of 0.001, to 15 places. The column "checks" contains the errors in (1) and in $\frac{1}{2}x = \sum_{k=0}^{\infty}(2k+1)J_{2k+1}$ in units of 10^{-15}. All values were computed to one place more. When the following digit was approximately 5, the last retained digit was increased by one unit or not, according to whether the actual value was slightly larger or slightly smaller than 5. In these cases the last retained digit has been crossed (/) or underlined (__), respectively. In part II the Taylor series for J_n of even order are given ($x=2, 3, 4, 5, 6$) (coefficients to 12 places). In part III the coefficients A_k in $J_n(x+h) = \sum_{k=0}^{\infty}A_k J_k(h)$ are given to 12 places for $n=2(2)30$, $x=7, 8, 9, 10$. Part IV gives a bibliography of text-books and tables of Bessel functions. [The author calls attention to the following correction: on p. 47, $J_5(6.93)$, read .35385 88367 2.]

S. C. van Veen (Delft).

Cambi, Enzo. **Complete elliptic integrals of complex Legendrian modulus.** J. Math. Physics 26, 234–245 (1948).

For real z a convenient method of computation of the complete elliptic integrals

$$K(z) = \int_0^{\frac{1}{2}\pi}(1-z\sin^2\varphi)^{-\frac{1}{2}}d\varphi, \quad E(z) = \int_0^{\frac{1}{2}\pi}(1-z\sin^2\varphi)^{\frac{1}{2}}d\varphi$$

is that of Gauss's arithmetico-geometrical mean (a.g.m.). When z is complex the computation of the complex a.g.m. scale is more tiresome, although the rapidity of convergence is very satisfactory. The evaluation of complex terms can be avoided in the case $|z| = 1$. For $z = -e^{ix} = e^{-iy}$ (x real, $y = \pi - x$) in the computation of $K'(z) = K(1-z)$, i.e., the a.g.m. $a_0 = 1$, $b_0 = k = e^{-\frac{1}{2}iy}$, all the terms of the scale which follow the first ones have the same phase.

The author has computed several tables for $|z| = 1$: (I) K' for $k^2 = -e^{ix} = e^{-iy}$ or K for $k^2 = 1 + e^{ix} = 1 - e^{-iy}$; (II) the same with K and K' interchanged; 10 decimals; $x = 0(.1)2$ in units of a right angle; Δ^2, Δ^4, Δ^8; (III) K and K' in the vicinity of $k^2 = 1$; $x = 1.9(.01)2$ in units of a right angle; modulus of K and K' to 10 decimals, phase of K to 7 decimals; (IV) for given $k^2 = -e^{ix}$, modulus k', $\Re(K/K')$, $|K/K'|$, $-\log_{10}|q'|$ ($q' = e^{-\pi K/K'}$) to 10 decimals; $x = 0(.1)2$ in units of a right angle; (V) modulus $k'K$ to 10 decimals; phase $k'K$, $\Re(k'K)$ and $\Im(k'K)$ to 7 decimals; (VI) $\Re e^{\frac{1}{2}iy}\{1 - E'/K'\}$ and $\Im e^{\frac{1}{2}iy}\{1 - E'/K'\}$ to 10 decimals. The integrals of the second kind can be expressed in terms of the a.g.m. scale by means of the relation

$$(K - E)/K = \frac{1}{2}\sum_{m=0}^{\infty}2^m c_m^2$$

$$(c_0^2 = k'^2, c_n^2 = a_n^2 - b_n^2, n \geq 1). \qquad S. C. van Veen \text{ (Delft)}.$$

Dwight, H. B. **Table of roots for natural frequencies in coaxial type cavities.** J. Math. Physics 27, 84–89 (1948).

Let $x_{n,s}$ and $x'_{n,s}$ denote the sth root of, respectively,

$$J_n(x)N_n(kx) - J_n(kx)N_n(x) = 0$$

and

$$J_n'(x)N_n'(kx) - J_n'(kx)N_n'(x) = 0,$$

where $J_n(x)$ and $N_n(x)$ are Bessel functions of the first and second kinds. Tables are given for $(k-1)x_{n,s}$ with $n=0(1)3$, $s=1(1)5$, and for $(k-1)x'_{n,s}$ with $n=1(1)3$, $s=2(1)6$, while $x'_{n,1}$ ($n=1, 2, 3$) is given directly ($x'_{0,s} = x_{1,s}$). The range of k is given by $k=1.0(0.1)1.6$, 1.8, 2.0(0.5)4.0, 5, 6(2)20, 25(5)50, though in one case k does not exceed 2.0. The number of decimals given decreases with increasing k from four to one.   *C. J. Bouwkamp* (Eindhoven).

Bose, P. K. **On recursion formulae, tables and Bessel function populations associated with the distribution of classical D^2-statistic.** Sankhyā 8, 235–248 (1947).

The distribution in question was found by R. C. Bose [Sankhyā 2, 143–154 (1936)]. It can be put in the form

$$P(L) = \int_0^P x^{q+1}\lambda^{-q}e^{-\frac{1}{2}(x^2+\lambda^2)}I_q(x\lambda)dx, \quad q = \frac{1}{2}p - 1,$$

where $2L^2 = \bar{n}pD_1^2 = \bar{n}pD^2 + 2p$, $2\lambda^2 = \bar{n}p\Delta^2$ and Δ^2 and D^2 are the population and estimated squared distances of two p-variate samples whose harmonic mean size is $\bar{n}$. Values of L are tabulated for $P = 0.99$, 0.95, 0.05, and 0.01, $p=1(1)10$ (that is, $q = -\frac{1}{2}(\frac{1}{2})\frac{7}{2}$) and $\lambda = 0.0(0.5)3.0(1)6$, 8, 12, 18, 24, 36, 54, 72, 108, 216, 432 to two decimals. Values of $e^{-x}I_0(x)$ and $e^{-x}I_1(x)$ are tabulated to six decimals from $x=16.08$ to $x=50.00$ at intervals usually multiples of 0.04 and not exceeding 0.32.   *J. W. Tukey* (Princeton, N. J.).

Rubbert, F. K. **Praktische Interpolation höherer Ordnung.** Z. Angew. Math. Mech. 28, 122–124 (1948).

Let the interval of the table be ω and $u = h/\omega$. Then the interpolation formula

$$y(x_0 + h) = (1-u)y_0 + uy_1 + \frac{1}{6}u(1-u)(y_0' - y_1')$$

with the error $\frac{1}{12}u(1-u)(1-2u)\delta^3 y_1$ is better than quadratic interpolation and useful for functions $y(x)$ where $y'(x)$ is also tabulated. The formula for inverse interpolation

$$x = x_0 + \frac{y-y_0}{y_1-y_0}\left\{1 + \frac{y_1-y_0}{y_2-y_1}\left(1 - 2\frac{y_1-y_0}{y_2-y_0}\right)\right\}$$

is quadratic and the formula $x = x_0 + Y_0/(Y_0+Y_1)$, where $Y_0 = (y-y_0)y'(x_0+\frac{1}{2}\omega)$, $Y_1 = (y_1-y)y'(x+\frac{1}{2}\omega)$, which may be used if $y'(x)$ is tabulated or easily computed, is more than quadratic. The inverse quadratic interpolation for $y = e^x$ is $x = x_0 + A(y-y_0)$ where $A^{-1} = (y-y_0) + (y_1-y)e^{-w/2}$.

E. Bodewig (The Hague).

Rubbert, F. K. **Zur Radizierung mit der Rechenmaschine.** Z. Angew. Math. Mech. **28**, 190–191 (1948).

Lo-Ho. **Construction of alignment nomogram from empirical data.** J. Franklin Inst. **245**, 227–244 (1948).

The method employed by M. Gorodskiĭ [Učenye Zapiski Moskov. Gos. Univ. Nomografiya **28**, 15–19 (1939); these Rev. **1**, 254] for constructing an approximate alignment chart for a tabulated relation between three variables $r = f(t, u)$ is presented except that the arbitrary r-scale, originally selected on a straight line, and the other two resulting scales, are not improved by successive approximations. The author discusses the error and shows how the two t-points from which the construction starts can be selected so that the diagram is exact for one arbitrary set of values in addition to those employed in the construction. The method is applied to the incomplete gamma function, to the elliptic integrals and to some physical data.

R. Church (Annapolis, Md.).

Pflanz, Erwin. **Über eine Verallgemeinerung des Verfahrens der Kombination von Newton'scher Methode und Regula falsi zur Auflösung einer Gleichung** $f(x)=0$. Z. Angew. Math. Mech. **28**, 114–122 (1948).

To determine the real root X of $f(x)=0$, $f(x)$ may be replaced by a function $p(x)$ which has the same values of the 0th, 1st, $\cdots$, nth derivatives as $f(x)$ at m points x_i near X. The author treats the case $m=2$ and determines two functions $p(x)$ which include the function $f(x)$ in the interval (x_1, x_2), thus generalizing the well-known device of Dandelin of combining Newton's method and the regula falsi. The zeros of the two $p(x)$ are bounds of X. The procedure converges.

E. Bodewig (The Hague).

Kincaid, W. M. **Solution of equations by interpolation.** Ann. Math. Statistics **19**, 207–219 (1948).

In the first part the author treats the solution of equations in one unknown by the well-known method of inverse interpolation, especially for the case that the first derivative is calculated. The second part treats simultaneous equations by means of Newton's formula or linear interpolation (rule of false position).

E. Bodewig (The Hague).

Willers, Fr. A. **Anschauliches zur Konvergenz des Iterationsverfahrens von Steffensen.** Z. Angew. Math. Mech. **28**, 125–126 (1948).

Steffensen [Skand. Aktuarietidskr. **16**, 64–72 (1933)] gave for the equation $x = f(x)$ the approximate solution $x = (x_1x_2 - x_2^2)/\Delta_2^2$, where $x_2 = f(x_1)$, $x_3 = f(x_2)$. The author gives a geometrical illustration of the convergence of the procedure.

E. Bodewig (The Hague).

Lin, Shih-Nge. **Numerical solution of complex roots of quartic equations.** J. Math. Physics **26**, 279–283 (1948). To factor

$$x^4 + A_3x^3 + A_2x^2 + A_1x + A_0 = (x^2 + a_1x + a_0)(x^2 + b_1x + b_0)$$

the author determines the parameters $\alpha = A_1/2A_0^{\frac{1}{2}}$, $\beta = A_2 - 2A_0^{\frac{1}{2}}$, $\gamma = A_3/2$. Then four cases arise: (I) $\beta = \alpha^2 = \gamma^2$; then the two factors are equal and $a_0 = b_0 = A_0^{\frac{1}{2}}$, $a_1 = b_1 = \gamma$; (II) $\beta < \alpha^2 = \gamma^2$; then $a_0 = b_0 = A_0^{\frac{1}{2}}$ and $a_1, b_1 = \gamma \pm (\gamma^2 - \beta)^{\frac{1}{2}}$; (III) $\beta < \alpha^2 \neq \gamma^2$; then a_i, b_i are determined by an iteration process; (IV) $\beta > \alpha^2$ or $\beta > \gamma^2$; then the author's method [same J. **22**, 60–77 (1943); these Rev. **5**, 49] is applied. When A_0 or A_2 is not positive, real roots are present. There are some slight misprints.

E. Bodewig (The Hague).

Melent'ev, P. V. **On the solution of equations of high degree.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 215–218 (1948). (Russian)

L'auteur donne une méthode pour le calcul numérique des racines imaginaires de l'équation

$$(1) \qquad F(z) = z^{2n} + A_1z^{2n-1} + A_2z^{2n-2} + \cdots + A_{2n} = 0$$

à coefficients réels. Soit $\alpha_0 + i\beta_0$ une telle racine, $\beta_0 i$ sera racine de l'équation $\phi(\zeta) = F(\zeta + \alpha_0) = \zeta^{2n} + B_1\zeta^{2n-1} + \cdots + B_{2n} = 0$, et $-\beta_0^2 = u_0$ sera racine commune des équations

$$P_n(u) = u^n + B_2u^{n-1} + B_4u^{n-2} + \cdots + B_{2n} = 0,$$

$$P_{n-1}(u) = B_1u^{n-1} + B_3u^{n-2} + \cdots + B_{2n-1} = 0.$$

Supposons que la racine $\alpha_0 + i\beta_0$ soit simple. En effectuant sur $P_n(u)$ et $P_{n-1}(u)$ l'opération du plus grand commun diviseur, désignons par $P_{n-2}(u), \cdots, P_1(u)$, P_0 les restes obtenus. Alors P_0 dépend seulement de α et on trouve de l'équation $P_0 = 0$ une valeur approchée de α_0. De l'équation $P_1(u) = r_1u + r_2 = 0$ on tire alors la valeur approchée de u_0, en supposant que α_0 est une racine simple de $P_0 = 0$. L'auteur donne un exemple numérique et des indications pour le cas où (1) a des racines multiples.

N. Obrechkoff.

Aparo, Enzo. **Applicazione di un nuovo metodo per la risoluzione numerica delle equazioni algebriche.** Bol. Soc. Portuguesa Mat. Sér. A. **1**, 49–57 (1948).

Sebastião e Silva [Portugaliae Math. **2**, 271–279 (1941); these Rev. **3**, 235] has given a method resembling that of Graeffe for solving algebraic equations. In the opinion of the reviewer the method is interesting but in practice much inferior to Graeffe's method, a point which was already emphasized by Sebastião e Silva himself. The author now gives a numerical example of the new method which illustrates this fact implicitly, since the amount of calculation is enormous.

E. Bodewig (The Hague).

Bodewig, E. **Bericht über die verschiedenen Methoden zur Lösung eines Systems linearer Gleichungen mit reellen Koeffizienten. IV, V.** Nederl. Akad. Wetensch., Proc. **51**, 53–64, 211–219 = Indagationes Math. **10**, 24–35, 82–90 (1948).

This paper continues the author's studies on methods of solution of systems of linear equations [same Proc. **50**, 930–941, 1104–1116, 1285–1295 = Indagationes Math. **9**, 441–452, 518–530, 611–621 (1947); these Rev. **9**, 250, 382]. This portion is devoted to analysis of various methods of iteration, and procedures for accelerating the convergence of the iteration process. The author shows that quadratic iteration is the most economical and considers the conditions for convergence. The methods of Kaczmarz and

Cimmino are shown to converge in all cases but too slowly for practical use. A third method, based on the expansion of the matrix in a power series in terms of the parameter, is discussed and compared with the above methods. The various methods that have been proposed to accelerate the convergence of the iteration are all shown to be too costly in labor to be of value. The author's final conclusion is that Gauss's method is essentially the best method available.
W. E. Milne (Corvallis, Ore.).

Andersen, Einar. Solution of great systems of normal equations together with an investigation of Andrae's dot-figure. An arithmetical-technical investigation. Mém. Inst. Géodésique Danemark [Geodætisk Instituts Skr.] (3) 11, 65 pp. (1947).

For the case of normal equations the paper gives a description of the usual Gauss (elimination) method and of the methods of Cholesky and Banachiewicz with an enumeration of the number of operations necessary, and of Seidel's method of iteration with historical remarks. The new result of the paper is that Cholesky's triangle equations (apart from proportional factors) are the same as Gauss's equations. Two examples with 7 and 35 equations, respectively, are solved by various methods. Finally the author discusses Andrae's device of the dot-figure in the practical solution of the equations of triangulation and concludes that its importance is rather doubtful.   *E. Bodewig.*

Laderman, Jack. The square root method for solving simultaneous linear equations. Math. Tables and Other Aids to Computation 3, 13–16 (1948).

The author describes the technique of the indicated method for solving a symmetrical system of linear equations. He attributes the method to T. Banachiewicz [1938]. In reality the method was invented 20 years earlier by Cholesky and was first described by Benoit [Bull. Géodésique 1924, 67–77]. The reviewer [cf. the second preceding review] proved that the method is inferior to the usual Gauss method as far as the number of operations is concerned. Andersen [see the preceding review] proved that the equations of both methods are the same [see also part V "Nachtrag," of the reviewer's paper, second preceding review]. The application of the method to an asymmetrical system $MX = N$ by solving the system $(M'M)X = M'N$ is regarded with suspicion by the author; in fact, in the report mentioned the reviewer proved that it requires more than twice as many operations as the Gauss method.
E. Bodewig (The Hague).

Cassinis, Gino. Risoluzione dei sistemi di equazioni algebriche lineari. Rend. Sem. Mat. Fis. Milano 17, 62–78 (1946).

The paper gives a brief description of iteration methods [the usual one and Seidel's], of the methods of Gauss, Banachiewicz, Cholesky, Boltz and of the method of orthogonalization. The latter is attributed to M. Sofia Roma [1945], although in reality it is the method of E. Schmidt [Rend. Circ. Mat. Palermo 25, 53–77 (1908)]. Cholesky's method is applied only to normal equations, although it is applicable to every symmetric equation. Here as in other papers the device of the "normalization" of an arbitrary system of equations in order to apply the methods established for normal equations is repeated. [This device, first given by von Mises and Pollaczek-Geiringer [Z. Angew. Math. Mech. 9, 58–77, 152–164 (1929)], seems to be a

technical fault. See von Neumann and Goldstine, Bull. Amer. Math. Soc. 53, 1021–1099 (1947); these Rev. 9, 471, and also the preceding review.] The comparison of the different methods by comparing the numbers of parameters to be computed is misleading, as the author states himself, since the amount of calculation of the parameters varies considerably with the method.   *E. Bodewig* (The Hague).

Mikeladze, Š. E. On the evaluation of determinants whose elements are polynomials. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 219–222 (1948). (Russian)

If $F(\lambda)$ is the value of a determinant of order n whose elements are polynomials in λ then $F(\lambda)$ is a polynomial of degree k and can be expressed in finite form by Taylor's series. The requisite coefficients $F^{(m)}(0)$ are found by the formula $F^{(m)}(0) = \sum_{\nu=0}^{k} A_{m\nu} \Delta^\nu F(0)$, where $\Delta^\nu F(0)$ is the νth difference obtained from $F(0), F(1), \cdots, F(k)$. The remainder of the paper supplies the numerical values of the coefficients $A_{m\nu}$ for $m = 1$ to 20, $\nu = m$ to 20.
W. E. Milne (Corvallis, Ore.).

Goruškin, V. I. Linear transformation of coordinates in the theory of electric machines and matrix calculus. Izvestiya Akad. Nauk SSSR. Otd. Tehn. Nauk 1948, 533–544 (1948). (Russian)

Ellis, Max E., and Riopelle, Arthur J. An efficient punched-card method of computing $\sum X$, $\sum X^2$, $\sum XY$, and higher moments. Psychometrika 13, 79–85 (1948).

Akuškii, I. Ya. Certain operational cycles of a tabulator, connected with the representation of numbers in the binary system. Doklady Akad. Nauk SSSR (N.S.) 59, 1521–1524 (1948). (Russian)

The author describes certain elaborate procedures by means of which an IBM tabulator may be used to multiply. One factor, the multiplier, is converted by hand into the binary system and a deck of cards is punched to represent this one number. A particular card of this deck is either x-punched or not according as the corresponding binary digit of the multiplier is 1 or 0. This deck is fed into the tabulator where it programs the appropriate doublings and accumulations to build up the product. No actual data as to the size of the numbers used or the time required are given. Applications are mentioned to the preparation of tables of factorials and powers without indicating the extent of such tables.   *D. H. Lehmer* (Berkeley, Calif.).

Akuškii, I. Ya. New methods for the calculation of a sum of products with a tabulator. Doklady Akad. Nauk SSSR (N.S.) 60, 5–8 (1948). (Russian)

The methods of the paper reviewed above are applied to the problem of calculating sums of products of as many as four factors using the tabulator alone. Applications are mentioned to the multiplication of matrices and vectors. No actual examples showing the feasibility of the methods are given.   *D. H. Lehmer* (Berkeley, Calif.).

✶**Goldstine, Herman H., and von Neumann, John.** Planning and Coding of Problems for an Electronic Computing Instrument. Report on the Mathematical and Logical Aspects of an Electronic Computing Instrument, Part II, Volume II. The Institute for Advanced Study, Princeton, N. J., 1948. iv+68 pp.

[This is a continuation of the reports reviewed in these Rev. 9, 208.] In this part the authors show how to code a

few typical problems in the evaluation of definite integrals, polynomial interpolation, meshing of two ordered sequences, and sorting a given set. In the last two cases they find that their proposed electronic computing instrument will be around 10 to 100 times faster than the classical IBM electromechanical sorters. *R. W. Hamming.*

Alt, Franz L. A Bell Telephone Laboratories' computing machine. II. Math. Tables and Other Aids to Computation **3**, 69–84 (1948).

This is the concluding article of a series of two describing the Bell Telephone Laboratories' computing system [for part I see the same vol., 1–13 (1948); these Rev. **9**, 307]. In this paper the author discusses the control, the machine's operational characteristics and finally a partial list of problems treated. In discussing the control he describes the more important orders the machine can obey. Under operational characteristics he considers topics such as reliability, the means used for checking as well as the methods employed for locating and remedying troubles and closes this subject with a very brief discussion of the coding problem for the machine. In the last chapter he enumerates a number of types of problems already solved on the Bell machines at Aberdeen. *H. H. Goldstine* (Princeton, N. J.).

Wilkes, M. V. The ENIAC—high-speed electronic calculating machine. Electronic Engrg. **19**, 104–108 (1947).

Mynall, D. J. Electrical analogue computing. I. Electronic Engrg. **19**, 178–180 (1947).
Mynall, D. J. Electrical analogue computing. II. Electro-mechanical multiplication, division and integration. Electronic Engrg. **19**, 214–217 (1947).
Mynall, D. J. Electrical analogue computing. III. Functional transformation. Electronic Engrg. **19**, 259–262 (1947).
Mynall, D. J. Electrical analogue computing. IV. Pure electronic systems. Electronic Engrg. **19**, 283–285 (1947).
Expository article.

Redheffer, Raymond. A machine for playing the game nim. Amer. Math. Monthly **55**, 343–349 (1948).

Steinhaus, H. Sur la cubature des troncs de bois. Colloquium Math. **1**, 23–28 (1947).

Approximate formulas for the volume of a truncated cone in terms of height h and the radii $r < R$ of the bases. For $r/R \leq \frac{1}{2}$, the error of $0.8h(r+R)^2$ is less than 2%, while that of $0.7818h(.94494r+1.05506R)^2$ is less than 0.6%. The last is best possible of its type. *J. W. Tukey.*

Ascoli, Guido. Un'osservazione sulle formule di quadratura. Boll. Un. Mat. Ital. (3) **2**, 212–216 (1947).

For empirical functions integration formulae of the type

$$\int_a^b f(x)dx = (b-a)\sum_1^k \lambda_r f(a+t_r(b-a)),$$
$$0 \leq t_1 < \cdots < t_k \leq 1, \ \lambda_r > 0,$$

are important. The author proves the following theorem. Let $f(x)$ be integrable in the sense of Lebesgue in every finite part of an open interval (A, B) and let the above relation hold for every a, b in the interval. Then $f(x)=0$ if $\sum \lambda_r \neq 1$. For $\sum \lambda_r = 1$ the above relation holds for all polynomials of degree not greater than s and only for them. Here s is the least integer for which $\sum \lambda_r t_r^{s+1} \neq 1/(s+2)$. *E. Bodewig* (The Hague).

Hartley, H. O., and Khamis, S. H. A numerical solution of the problem of moments. Biometrika **34**, 340–351 (1947).

Application of finite difference methods to the calculation of $F(a+ih)$, where $F(u) = \int_a^u f(x)dx$, and where $\mu_r = \int x^r f(x)dx$ is known for a finite number of r's. Once-for-all inversion of a matrix leads to explicit linear expressions for $F(a+ih)$, $i=1, 2, \cdots, R+1$, in terms of μ_r, $r=0, 1, 2, \cdots, R$, plus error terms. The present form of these error terms is not very manageable, but their discussion is promised in a later paper. When $f(x)$ is nonzero on a half-line or everywhere, limiting methods are required.

Application to $I_a(8, 6)$ for $x=0(\frac{1}{4})1$ gives maximum error of 2.2×10^{-4}. Application to chi-square on 10 degrees of freedom for $x=-.915(1.0)5.085$ gives maximum error of 3.6×10^{-4}. Application to unit normal for $x=-4(1)4$ gives maximum error of 1.5×10^{-5}. Application to $t^{\frac{1}{2}}$ on 10 degrees of freedom for $t^{\frac{1}{2}}=0(0.6)2.4$ gives maximum error of 3.5×10^{-3}. Thus use of $t^{\frac{1}{2}}$ to improve convergence was not sufficient for high accuracy. The absolute moments of t required were obtained by interpolation of the logarithmic moment function, whose accuracy will be discussed in a later paper. *J. W. Tukey* (Princeton, N. J.).

Beard, R. E. Some notes on approximate product-integration. J. Inst. Actuar. **73**, 356–403; discussion, 404–416 (1947).

The paper develops further the method of Perks for evaluating the integral

$$\int_a^b f(x)\varphi(x)dx = [a_1 f(x_1) + \cdots + a_n f(x_n)]\int_a^b \varphi(x)dx$$

by elementary mathematics. It is concerned with the general solution of the above equation for several n's and its expression in terms of moment functions of φ. Thus the method is useful when φ is an empirical function defined only by its moments. The question is treated in four cases: (i) when the a_r and x_r have to be determined, (ii) when all $a_r = 1/n$ are prescribed and only the x_r are unknown, (iii) when the a_r are prescribed otherwise, (iv) when the x_r are prescribed. The formulae are applied to special forms of $\varphi(x)$: $\varphi=1$, $\varphi=\exp(-cx)$, $\varphi=\exp(-ax^2)$, $\varphi=x^m \exp(-cx)$, $\varphi=(1-x)^m(1-x)^r$. Tables are added for the practical use of the formulae, especially when φ is a Pearson distribution function. *E. Bodewig* (The Hague).

Bickley, W. G. Finite difference formulae for the square lattice. Quart. J. Mech. Appl. Math. **1**, 35–42 (1948).

If $f(x, y)$ is a function of two variables the author gives expressions for the partial derivatives of $f(x, y)$ and particularly for the quantities $\nabla^2 f$, $\nabla^4 f$, $\iint f(x, y)dxdy$, in terms of values of $f(x, y)$ at certain nodes of a square lattice in the (x, y)-plane. Such expressions are, of course, well known and the principal contribution of the paper lies in its analysis of the error terms and in the selection of optimum forms. *W. E. Milne* (Corvallis, Ore.).

Inzinger, R. Zur graphischen Integration linearer Differentialgleichungen mit konstanten Koeffizienten. Österreich. Ing.-Arch. **1**, 410–420 (1947).

Es sei P ein Punkt der orientierten Kurve k, t der Tangentialspeer von k in P und $t_{1\alpha}$ jener durch P gehende Speer, der mit t den Winkel α einschliesst. Bewegt sich nun P auf k, dann umhüllen für festes α die Speere $t_{1\alpha}$ eine orientierte Kurve $k_{1\alpha}$, die man als die α-Evolutoïde von k

bezeichnet. Für $\alpha = \frac{1}{2}\pi$ ergibt sich als Sonderfall die Evolute k_1 von k. Das von E. Meissner angegebene als Linienbildverfahren bekannte Integrationsverfahren besteht im wesentlichen darin, dass die durch eine gegebene Differentialgleichung n-ter Ordnung festgelegte Beziehung zwischen der gesuchten Funktion und ihren ersten n Ableitungen als Beziehung zwischen einer Kurve k und ihren ersten n Evoluten gedeutet werden kann [Schweiz. Bauztg. **98**, 287–290, 333–335 (1931); **99**, 27–30, 41–44, 67–69 (1932)]. Dieses Verfahren wird vornehmlich zur graphischen Integration von Differentialgleichungen zweiter Ordnung angewendet. Der Verf. betrachtet an Stelle der Evolutenbildung die α-Evolutoïdenbildung. Dann ist der Operator D durch den linearen Differentialoperator $D_{1\alpha} = \cos \alpha + D \sin \alpha$ zu ersetzen. So gelangt er zu einer Verallgemeinerung des Meissnerschen Verfahrens, dass sofort auf Differentialgleichungen von beliebig hoher Ordnung anwendbar ist. Durch die Polarität am Einheitskreis um den Ursprung wird aus dem Meissnerschen Verfahren das Orthopolarenverfahren von R. Grammel erhalten.

Es ergibt sich, dass an die Stelle der Grammelschen Orthopolarenbildung eine Konstruktion treten muss, die sinngemäss als α-Isopolarenbildung bezeichnet werden kann.

S. C. van Veen (Delft).

Duncan, W. J. Technique of the step-by-step integration of ordinary differential equations. Coll. Aeronaut. Cranfield. Rep. no. 4, 24 pp. (1947).

Duncan, W. J. Technique of the step-by-step integration of ordinary differential equations. Philos. Mag. (7) **39**, 493–509 (1948).

If the number of steps in any step-by-step process of solving a differential equation over a fixed range is n, the author expands the error in a series of powers of $1/n$ and calls the exponent of the lowest power actually present the "index" of the process. If the index is known, he shows how by using two or three different values of n the amount of the correction can be estimated. The simple cases $n=1$ and $n=2$ are illustrated by numerical examples. The last part of the paper presents a classification and discussion of various numerical methods of solving differential equations.

W. E. Milne (Corvallis, Ore.).

Duncan, W. J. Assessment of errors in approximate solutions of differential equations. Coll. Aeronaut. Cranfield. Rep. no. 13, 9 pp. (1947).

The paper first of all distinguishes between (a) the fixing of rigid upper and lower bounds to the error, and (b) the estimation, more or less closely, of the error. In part 1 it is shown that if an ordinary or partial linear differential equation with linear homogeneous boundary conditions has a Green's function which is one-signed then it is possible to assign rigid bounds to the error of an approximate solution which satisfies the boundary conditions. In part 2 it is shown that the errors in the step-by-step solution of an ordinary differential equation may be estimated by comparing the results for two or more integrations using different step-intervals [cf. the preceding review]. Part 3 considers the estimation of the errors in approximate solutions of linear problems when an approximation to the Green's function is known. *W. E. Milne* (Corvallis, Ore.).

Meksyn, D. The laminar boundary-layer equations. II. Integration of non-linear ordinary differential equations. Proc. Roy. Soc. London. Ser. A. **192**, 567–575 (1948).

[For paper I see the same vol., 545–567 (1948); these Rev. **9**, 632.] In the first part of this paper the equation $f''' + ff'' = \lambda(1 - f'^2)$ is discussed, where the parameter λ is usually known, except in the case when it is required to find the value of λ so that the integral may satisfy the additional condition $f''(0) = 0$ (this case is associated with the separation point in hydrodynamics). The boundary conditions are: $f = f' = 0$ at $x = 0$; $f'' = 0$, $f' = 1$ at $x = \infty$. The integration is done by successive approximations. In the simplest case: $f''(0) = 0$, to find λ, the first approximation gives $\lambda = 0.213$. A second approximation may be obtained by means of the trial $f = \sum_{k=0}^{\infty} a_{1+4k} x^{1+4k}/(3+4k)!$. This gives $\lambda = 0.197$. The correct value $\lambda = 0.199$ was obtained by Hartree [Proc. Cambridge Phil. Soc. **33**, 223–239 (1937)] by numerical integration. The second part deals with the approximate integration of a more general equation of the type

$$ f''' + ff'' = \lambda(1 - f'^2) + \epsilon\left\{ af'' + \int_x^\infty (bff'' + cxf'f'')dx \right\}, $$

where ϵ, a, b and c are known parameters, and ϵ is very small of the order of .01. No attempt is made to give a rigorous discussion of the integration; the solution is pursued only in so far as it is needed in the hydrodynamical problem.

S. C. van Veen (Delft).

Ragazzini, John R., Randall, Robert H., and **Russell, Frederick A.** Analysis of problems in dynamics by electronic circuits. Proc. I.R.E. **35**, 444–452 (1947).

Yuškov, P. P. On the application of triangular nets to the numerical integration of the equation of heat conduction. Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 223–226 (1948). (Russian)

Fourier's heat equation $\partial u/\partial t = a^2 \nabla^2 u$ is put in the form of a difference equation by setting $\partial u/\partial t = (U_{0i+1} - U_{0i})/l$, $t = il$, and replacing $\nabla^2 u$ by

$$ \tfrac{2}{3}h^{-2}(U_{1i} + U_{2i} + U_{3i} + U_{4i} + U_{5i} + U_{6i} - 6U_{0i}), $$

where $U_{1i}, \cdots, U_{6i}$ are the values of u at the vertices of a regular hexagon in the triangular net in the (x, y)-plane and U_{0i} is the value at the center. By choosing $l = h^2/8a^2$ the equation becomes

$$ U_{0i+1} = \tfrac{1}{12}(U_{1i} + U_{2i} + U_{3i} + U_{4i} + U_{5i} + U_{6i} + 6U_{0i}). $$

The initial temperatures for $i = 0$ together with the boundary values are known, and by successive applications of the formula the values of u_{ki} are computed for any i. A numerical example illustrates the process. *W. E. Milne.*

Bückner, Hans. A special method of successive approximations for Fredholm integral equations. Duke Math. J. **15**, 197–206 (1948).

This paper deals with the Fredholm integral equation

$$ y(s) = f(s) + \lambda \int_a^b K(s, t)y(t)dt $$

with continuous $K(s, t)$ and $f(s)$. With a given continuous function u_0 and a given fixed number θ the sequence $u_0(s), u_1(s), \cdots$ is computed by

$$ u_{n+1}(s) = \theta u_n(s) + (1-\theta)\lambda \int_a^b K(s, t)u_n(t)dt + (1-\theta)f(s). $$

If this sequence converges for a given pair of numbers λ, θ and for every continuous function u_0, the convergence is called total for the pair (λ, θ). The author obtains the following set of necessary conditions for total convergence of the sequence $u_n(s)$:

(a) $$|\theta + (1-\theta)\lambda_k^{-1}\lambda| < 1, \qquad k = 1, 2, \cdots,$$

where $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of the kernel $K(s, t)$, $|\lambda_i| \leq |\lambda_k|$ for $i < k$. In the case of a degenerate kernel total convergence can only take place for (b) $|\theta| < 1$. It is shown, furthermore, that the conditions (a) and (b) are also sufficient. The author states that this generalized approximation method (in the convenient case $\theta = 0$) seems to have been first proposed by G. Wiarda [Integralgleichungen, Leipzig, 1930], who obtained the necessary and sufficient conditions (a) and (b) for the case that $K(s, t)$ is symmetric and positive definite.   *S. C. van Veen* (Delft).

Carrier, G. F. On the determination of the eigenfunctions of Fredholm equations. J. Math. Physics 27, 82–83 (1948).

The author presents an extension of the usual iteration procedure for the solution of the homogeneous Fredholm integral equation with symmetric kernel which improves the convergence of the iteration. The modified process uses the rough approximations obtained for the eigenvalues λ_2,

λ_3, etc., rather than the successive approximations to λ_1, to secure improved values for λ_1 and for the eigenfunction φ_1. The theory is illustrated by a numerical example.
                          W. E. Milne (Corvallis, Ore.).

Meurers, Joseph. Fehlertrennung durch Wahrscheinlichkeitsbetrachtung. Z. Angew. Math. Mech. 28, 183–186 (1948).

Zwinggi, E. Eine Näherungsformel für die Prämie der Invalidenversicherung. Experientia 4, 218–219 (1948).

Giuliano, Salvatore. Sulla capitalizzazione a due tassi variabili. Matematiche, Catania 1, 21–29 (1945).

Tortorici, Paolo. Sulla determinazione del tasso d'interesse nelle annualità differite e non differite. Matematiche, Catania 2, 25–40 (1946).

✶**Ruffet, Jean.** L'Aplatissement Terrestre Calculé en Seconde Approximation. Thesis, University of Geneva, 1942. 63 pp.

Trombetti, Carlo. Sulla variazione delle coordinate geografiche dei vertici di una triangolazione per il cambio dei parametri dell'ellissoide di riferimento. Boll. Geodet. Ist. Geograf. Mil. 7, 47–66 (1948).

ASTRONOMY

Demetrescu, G. Sur la première approximation dans le calcul d'une orbite par la méthode de Gauss. Acad. Roum. Bull. Sect. Sci. 28, 642–644 (1946).

Sinding, Erik. On the systematic changes of the eccentricities of nearly parabolic orbits. Danske Vid. Selsk. Mat.-Fys. Medd. 24, no. 16, 8 pp. (1948).

Pierucci, Mariano. Una correzione relativistica della legge di Newton e la tendenza delle orbite planetarie alla circolarità e alla complanarità. Atti Soc. Nat. Mat. Modena 78, 1–4 (1947).

Carathéodory, C. Über die Integration der Differentialgleichungen der Keplerschen Planetenbewegung. S.-B. Math.-Nat. Abt. Bayer. Akad. Wiss. 1945/46, 57–76 (1947).

A discussion of the two-body problem for the Newtonian inverse square law in n-dimensional space is expressed in terms of the invariants $\mathbf{r} \cdot \mathbf{r}$, $\mathbf{v} \cdot \mathbf{v}$, and $\mathbf{r} \cdot \mathbf{v}$ and so does not differ materially from the treatment by standard vector methods [e.g., L. Brand, Vectorial Mechanics, Wiley, New York, 1930, sections 177, 182].   *P. Franklin.*

Zagar, Francesco. Ricerche dinamiche sopra i sistemi binari stretti. Rend. Sem. Mat. Fis. Milano 16, 100–142 (1942).

L'auteur présente un examen critique des différentes recherches sur le problème dynamique des systèmes binaires serrés. Il est bien connu que pour ces systèmes on ne peut pas admettre la forme sphérique des composantes et d'autre part on ne peut pas négliger le carré du rapport des dimensions des composantes à la distance des centres de masses des composantes mêmes. Les difficultés du problème sont encore plus graves si l'on passe de l'hypothèse de la rigidité des composantes à celle de la déformabilité. Les résultats

obtenus jusqu'ici sur le problème sont remarquables, mais on doit dire, après l'exposition de Zagar, qu'il est désirable une solution interprétant plus fidèlement la complexité des phénomènes observés sur la base d'une représentation mathématique plus conforme à la réalité physique.
                          G. Lampariello (Messina).

Sokolov, Yu. D. On the trajectories of the rejection to infinity of three material points moving under the influence of their mutual interaction. Doklady Akad. Nauk SSSR (N.S.) 58, 539–542 (1947). (Russian)

The author considers the motion of three particles P_i of masses m_i ($i = 0, 1, 2$) which attract or repel each other, the interaction between P_i and P_j having magnitude $m_i m_j |f(r_k)|$ ($i \neq j \neq k$; $r_k = P_i P_j$) and representing an attraction or repulsion according as f is positive or negative. It is assumed that f is analytic for positive r, continuous for $r = 0$, and such that, as r becomes infinite, $\lim r^{1-2\alpha} f(r) = 2\alpha$, where $\alpha > \frac{1}{2}$. The problem then considered is the behaviour of a solution, defined for $0 \leq t < t_1$, which fails to be regular at t_1. It is shown that as t approaches t_1 the moment of inertia I of the system about its center of mass either approaches a finite value or becomes infinite. It is shown that in the latter case necessarily $\alpha > 1$.

A special study is made of the case of planar motion. The ratio r_i/I is denoted by p_i and the smallest value p_i by p_m. It is then asserted that if, for $t \to t_1$, I becomes infinite but (1) $\inf p_m > 0$ or (2) $\lim p_m = 0$, then only three cases can arise: (a) $\lim p_0 = \lim p_1 = \lim p_2 = G_1(m_0, m_1, m_2)$, (b) $\lim p_2 = G_2(m_0, m_1, m_2, q)$, $\lim p_0/p_2 = q$, $\lim p_1/p_2 = 1 + q$, (c) $\lim p_0 = 0$, $\lim p_1 = \lim p_2 = G_2(m_0, m_1, m_2)$, where it is assumed that $p_0 = p_m$ for t sufficiently close to t_1, G_1, G_2, G_3 denote certain simple algebraic expressions, and q denotes a positive root of a certain algebraic equation. The differential equations are reduced to four first order equations, by

an appropriate change of variables, and it is stated that, under appropriate assumptions, the methods of Bol and Cotton can be applied to obtain asymptotic representations of the solutions for large values of I.   *W. Kaplan.*

Lahaye, Edmond. La régularisation du mouvement dans le problème des N corps et les itérations intégrales convergentes. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **33**, 652–671 (1947).

The author uses the familiar regularizing transformation [cf. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton University Press, 1941, pp. 334–338; these Rev. 3, 215] for binary collisions in the n-body problem. A fixed time interval in which only binary collisions occur is then considered and it is shown that in this interval the solutions can be obtained by a Picard iteration process.
                                           W. Kaplan (Ann Arbor, Mich.).

Coutrez, Raymond. Contribution à la théorie de Lindblad sur le mouvement des masses dans la dynamique des systèmes stellaires. Stockholms Observatoriums Annaler **15**, no. 3, 38 pp. (1947).

This paper represents an extension of Lindblad's earlier studies on the dynamical equilibrium of stellar systems starting, not from statistical considerations, but from equations governing the mean motions of mass particles in stellar systems and their perturbations by neighboring particles. The perturbations are, however, regarded in the present investigation to be small enough for their squares and higher powers to be negligible. The investigation deals with three-dimensional motions and takes explicit account of the effects of differential rotation. A relation is established showing the effect of time upon the symmetry of large-scale motions on the periphery of highly flattened stellar systems; this relation should permit one to determine the motion of the loci at which ejection of matter can take place, at the boundary of such systems, by way of orbits which are open, or are

asymptotic to circles of very large radii. The stability of motion in three dimensions is examined, and conditions established for which the ejection may actually take place.
                                           Z. Kopal (Cambridge, Mass.).

Pecker, Jean-Claude. Sur une méthode d'intégration des équations d'équilibre des atmosphères stellaires. C. R. Acad. Sci. Paris **226**, 561–563 (1948).

This extends an earlier discussion of Kourganoff [same C. R. **225**, 491–493 (1947); these Rev. 9, 190] to the solution of the equation of transfer which includes scattering and absorption.   *S. Chandrasekhar* (Williams Bay, Wis.).

Parvulesco, Const. Sur les pulsations des systèmes stellaires. Acad. Roum. Bull. Sect. Sci. **25**, 8–14 (1943).

The author distinguishes between two phases in the evolution of a galaxy. In the first phase a nonrotating galaxy left by itself is shown to have a spherical shape. The most massive stars will have the greatest concentration about the center. The author attempts to demonstrate that close encounters and actual collisions will tend to increase the degree of central condensation. The second phase begins when an encounter between two galaxies takes place. The tidal forces between the two galaxies result in an elongation of the perturbed galaxy in the direction of the perturbing galaxy. At diametrically opposite points, the tidal forces from the passing galaxy, combined with the effect of the centrifugal force of rotation, lead to instability, with as a result material flowing outward along spiral arms. Certain gyroscopic effects tend to speed up the dissolution of the system as a spiral.

The paper is in the nature of a brief and general exploratory treatment. Without a more extensive analysis, one cannot accept the conclusions as demonstrated. No attempts are made to relate the new approach to those suggested by Lindblad and other authors and there is no indication as to how the author expects to be able to justify his conclusions on observational grounds.   *B. J. Bok.*

RELATIVITY

Walker, A. G. Foundations of relativity. I, II. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 319–335 (1948).

The author argues that a theory of relativity can be either physical, mathematical or logical. It is physical if some of the elementary objects and relations are concepts derived from the external world and if certain of their properties are assumed as physically obvious; mathematical if the elementary objects, etc., are defined as mathematical symbols and relations and if the subsequent theorems are deduced mathematically from the definitions; and logical if certain terms are undefined and if the theory is developed strictly deductively from an explicit set of axioms and definitions. The construction of a logical theory is not merely of academic interest, for it is in effect an analysis of previous theories. Newtonian mechanics is physical, general relativity mathematical, kinematical relativity again physical. In this paper the author begins a theory of cosmology deduced from a logical basis consisting of four undefined elements and relations, namely (i) instant, (ii) particle class of instants, (iii) temporal order and (iv) signal correspondence. The axioms are classified as "temporal," "signal" and "fundamental." He assumes only one "observer," and supposes that all axioms, definitions and "experiments" are expressed in terms of primitive observations made by this observer.

Part I is devoted to the study of temporal order, but no assumption is made that temporal order can be adequately described by the continuum of real numbers: instead, the author begins with a general type of order and deduces ordinal-linearity from the fundamental axioms. In part II he postulates and examines certain "fundamental particles," which correspond to the fundamental particles in kinematical relativity and to the particles in general relativity which are at rest relative to the matter near them.

No brief summary of this paper could give more than a bare indication of the author's line of thought, and that is all that the foregoing abstract is intended to do. The paper perhaps represents the most profound and ambitious attempt made in recent years to analyse the foundations of relativity, not excepting the work done by various authors in connection with Milne's theory.   *H. S. Ruse.*

Moghe, D. N. On the theory of a system of receding particles having a tendency to approach the central mass. Proc. Nat. Inst. Sci. India **6**, 225–235 (1940).

The object of this paper is to extend Milne's kinematical theory to a system of receding particles in a slightly inhomogeneous space with a dense cluster. The notation is that used by Milne and the modified equations of motion are

$dV/dt = (P - Vt)(Y/X)G(\xi) \pm YZH(X, \xi)$. The consequences of these equations are discussed at first generally and then particularly when $H(X, \xi) = KX^{-2}G(\xi)$ where K is a constant.

There appears to be some serious confusion between vector and scalar quantities. In the equations of motion, for example, P and V are vectors, t, X, Y, Z, $G(\xi)$ are scalars and it appears subsequently that H is also a scalar, as also is K in the restricted case. It is not clear to what extent, if any, the conclusions of the paper are invalidated by this confusion. *A. G. Walker* (Sheffield).

Fourès-Bruhat, Yvonne. Sur l'intégration du problème des conditions initiales en mécanique relativiste. C. R. Acad. Sci. Paris **226**, 1071–1073 (1948).

The author takes $ds^2 = \omega_0^2 - \sum \omega_i^2$, $\omega_0 = rdx^0$, $\omega_i = a_{ik}dx^k + \lambda_i dx^0$, and writes down the Einstein equations. She then states that she proposes to find, given on a hypersurface the components of the energy tensor with respect to any three coordinate-lines, what it is necessary to know of the surface itself, and of the position of the coordinate-lines with respect to the geometrical elements of the surface, in order that the Einstein equations should determine, on the one hand the $\overline{ds}^2$ of the surface $(\overline{ds}^2 = \sum \bar{\omega}_i^2, \bar{\omega}_i = a_{ik}dx^k)$, and on the other hand the ds^2 of a corresponding universe. She outlines a method by which this may be done, using a moving frame (repère mobile) of which the three axes tangent to the surface are in the direction of the tangents to the lines of curvature at the corresponding point. *H. S. Ruse.*

Hill, E. L. On the formal extension of Dirac's equation under continuous transformation groups. Physical Rev. (2) **73**, 910–915 (1948).

Let G be a continuous group of transformations of the space-time variables $(x_1, \cdots, x_4)$, having n essential parameters. The group may be defined by a set of n basis operators

$$X_\alpha = \xi_\alpha{}^k(x)p_k \quad (\alpha = 1, \cdots, n), \quad p_k = -i\hbar\partial/\partial x_k.$$

The author considers a generalised Dirac operator

$$M = \beta^\alpha \xi_\alpha{}^k(x)[p_k - \epsilon A_k/c] + im_0c,$$

where A_k is a 4-vector and ϵ, m, c constants. The problem is to find necessary and sufficient conditions on the properties of the operators β^α for M to be formally invariant under the group G. The conditions are found to be a certain set of differential equations, linear and homogeneous in the β^α. For M to be utilisable in a wave-equation $M\Psi = 0$, it is further necessary that the transformations of the β^α under G should be of the form $\beta'^\alpha = V\beta^\alpha V^{-1}$.

It is shown that if G is the inhomogeneous Lorentz group then the β^α have a continuous range of eigenvalues and so cannot admit a finite algebraic representation. The theory was developed in order to treat just this case, which arose in earlier work of the author [same Rev. (2) **72**, 143–149 (1947); these Rev. **9**, 107]. *F. J. Dyson.*

Reulos, René. Champ électromagnétique et champ d'accélération. Cahiers de Physique nos. 31–32, 83–91 (1948).

L'auteur mentionne dans son résumé: "Cette étude a pour but de montrer l'analogie que présente le mouvement des masses matérielles dans un champ accéléré, avec le mouvement des charges électriques dans un système absolu (galiléen) où règne un champ électromagnétique. En particulier, le champ de rotation produit la force de Coriolis analogue à la force de Lorentz en électromagnétisme. Cette analogie peut être poursuivie jusque dans la forme même des équations que vérifient les champs: les uns et les autres dérivent d'un potentiel scalaire et d'un potentiel vecteur au

moyen de formules identiques. Le théorème de Larmor n'est que l'expression quantitative de cette analogie, qui en fournit une démonstration simple et intuitive."

En effet, il résulte pour la force qui agit sur un point matériel en mouvement l'expression $\mathbf{F} = m(\mathbf{g} + \mathbf{v} \times \mathbf{R})$ et pour la force de Lorentz $\mathbf{f}$ exercée sur la charge électrique e animée d'une vitesse $\mathbf{v}$ dans un champ électrique $\mathbf{E}$ et un champ magnétique $\mathbf{H}$, $\mathbf{f} = e(\mathbf{E} + \mathbf{v} \times \mathbf{H})$, dont l'analogie est frappante. L'auteur recherche la cause de cette analogie. En outre il obtient les résultats suivants. Le mouvement relatif d'un point matériel dans un champ gravifique et le mouvement absolu d'un corpuscle électrisé obéissent aux mêmes lois, qui dérivent des mêmes équations. Le mouvement d'un électron dans un champ magnétique $\mathbf{H}$ est le même qu'en l'absence du champ, à condition de l'évaluer dans un système de référence tournant a la vitesse Ω définie par la formule $\Omega = -\frac{1}{2}(e/m)\mathbf{H}$; Ω est la précession de Larmor. A titre d'application, le gyroscope est assimilé à un courant électrique circulaire et cette manière de voir permet d'en retrouver aisément les principales propriétés. *M. Pinl* (Cologne).

Klein, O. On a case of radiation equilibrium in general relativity theory and its bearing on the early stage of stellar evolution. Ark. Mat. Astr. Fys. **34A**, no. 19, 11 pp. (1948).

Recently G. Beskow and L. Treffenberg [same vol., no. 17 (1948)] examined a thermal and gravitational equilibrium state of radiation and matter (mostly radiation), and deduced results about the distribution of different nuclei which are in excellent agreement with observation. The author points out that a homogeneous universe could not serve as a model for the equilibrium state of Beskow and Treffenberg: an Einstein universe would be too small, whereas an expanding universe would not even approximate equilibrium conditions. The author therefore turns to the assumption that the equilibrium states of Beskow and Treffenberg are localized around individual stars. As a first step towards a relativistic treatment of this model, spherically symmetric solutions of the gravitational equations are investigated which represent unordered radiation. Such radiation stars may have arbitrary mass density at the center, the density decreasing at large distances from the center by an inverse square law. These radiation stars have a behaviour very similar to that of Emden gas spheres. A radiation star with a central temperature of 10^{10} degrees corresponds to the conditions considered by Beskow and Treffenberg. Although the author's solution is static there are good reasons for supposing the solution to be unstable in the long run. *A. Schild* (Toronto, Ont.).

Sibata, Takasi. On space which has the homogeneous property for observation systems. J. Sci. Hirosima Univ. Ser. A. **11**, 231–243 (1942).

Itimaru, Kusuo. Cosmology in terms of wave geometry. X. Observers on the nebulae. J. Sci. Hirosima Univ. Ser. A. **11**, 245–253 (1942).

Sibata, Takasi, and Sakuma, Kiyosi. Cosmology in terms of wave geometry. XI. The solar system as a local irregularity in the universe. J. Sci. Hirosima Univ. Ser. A. **11**, 255–272 (1942).

Sakuma, Kiyosi, and Sibata, Takasi. Generalized geodesic lines and equation of motion in wave geometry. J. Sci. Hirosima Univ. Ser. A. **11**, 273–291 (1942).

These papers constitute numbers 48 to 51 of a series of papers on wave geometry published by the Hirosima Uni-

versity group [cf. the same J. 10, 151–156, 173–214, 247–252 (1940); these Rev. 2, 208]. The first two papers are classical, being essentially independent of wave geometry. Sibata examines conditions for a relativistic universe to be stationary and homogeneous; he arrives at the well-known result that the only such universes are the Minkowski, the Einstein and the de Sitter universes. Itimaru shows that the particles (nebulae) of a de Sitter universe are equivalent in the sense of Milne's kinematical relativity.

In the third paper, Sibata and Sakuma examine the problem of planetary motion on the basis of wave geometry. In non-Newtonian approximation, they find, besides the relativistic advance of perihelion, a change in time of the eccentricity and latus rectum of the orbit.

Sakuma, in the fourth paper, considers a variational principle $\delta \int f(S)ds = 0$ which generalizes the concept of geodesics satisfying $\delta \int ds = 0$. Here $S = \{(\psi^\dagger A \psi)^2 + (\psi^\dagger A \gamma_5 \psi)^2\}^{\frac{1}{2}}$ [see the reference above]. The following results are obtained. In general, the motions given by $\delta \int f(S)ds = 0$ are along geodesics in the Einstein or de Sitter universes. In the special case $f = 1/S$, the motions are those of particles in a unified field of gravitation and electromagnetism of the Born-Infeld type. If $f = S$, the motions are those of particles in a dissipative medium.   A. Schild (Toronto, Ont.).

MECHANICS

Vasilesco Karpen, N. Sur un cas curieux d'équilibre. Acad. Roum. Bull. Sect. Sci. 25, 4–7 (1943).

Cârstoiu, I. Sur un cas curieux d'équilibre. Acad. Roum. Bull. Sect. Sci. 27, 421–423 (1947).

Germani, D. Démonstration élémentaire du théorème d'équilibre de Mr. N. Vasilesco Karpen. Acad. Roum. Bull. Sect. Sci. 28, 1–3 (1945).

Un solide, de forme quelconque, soumis en chaque point de sa surface, à une pression proportionnelle à la courbure moyenne au point considéré, est en équilibre.

O. Bottema (Delft).

Cârstoiu, I. Sur le mouvement d'un solide rigide. Acad. Roum. Bull. Sect. Sci. 29, 83–85 (1946).

L'accélération d'un point du solide, fixé par le vecteur r, sera $\mathbf{a} = \dot{\mathbf{v}}_0 + \dot{\omega} \times \mathbf{r} + \operatorname{grad} \varphi$, où

$$\varphi(x, y, z) = -(q^2 + r^2)x^2 - (r^2 + p^2)y^2 - (p^2 + q^2)z^2 + 2qryz \\ + 2rpzx + 2pqxy.$$

L'auteur donne une interprétation géométrique des invariants de cette forme quadratique.   O. Bottema (Delft).

Grammel, R. Kinetisch unbestimmte Systeme. Z. Angew. Math. Mech. 24, 215–223 (1944).

The paper is concerned with an extension of the concept of static indeterminacy. A truss, for instance, is called statically indeterminate when the forces in its members are not uniquely determined by considering the members of the truss as rigid bodies and using the equations of stereostatics. Similarly, the author speaks of kinetic indeterminacy if the motion of a system is not uniquely determined by considering its components as rigid bodies and using the equations of stereo-kinetics. Three kinds of kinetically indeterminate systems are defined and illustrated by examples.   W. Prager (Providence, R. I.).

Čebyšev, P. L. Theory of the mechanisms known as parallelograms. Uspehi Matem. Nauk (N.S.) 1, no. 2(12), 12–37 (1946). (Russian)

The original paper appeared in French in Mémoires Présentés à l'Académie des Sciences de St.-Pétersbourg par Divers Savants 7, 539–568 (1854); this translation was first published in the Russian edition of the author's collected works, v. I, St. Petersburg, 1899.

Hoch, Hans. Das physikalische Pendel im radialsymmetrischen Schwerefeld der Erde. Z. Angew. Math. Mech. 24, 240–243 (1944).

The author considers a rigid body, of arbitrary form and size, which is free to rotate about a fixed horizontal axis in the earth's gravitational field. Assuming that the linear dimensions of the body are small compared with the earth's radius, he takes approximate account of the variations in the intensity and direction of the gravitational field. Most of the discussion relates to the dependences of the number and the stabilities of the equilibrium configurations upon certain parameters relating to the form and size of the body, and to the frequencies of small oscillations about these configurations. The following is a typical result. If the centroid of the body lies on the axis of rotation, the frequency of any such small oscillation depends upon the form, but not upon the size, of the body.   L. A. MacColl.

Genty, Robert. Sur les problèmes de l'évasion hors de l'attraction terrestre et de la gravitation autour de la Terre. C. R. Acad. Sci. Paris 226, 1510–1512 (1948).

For rockets, the familiar hypothesis of ballisticians concerning acceleration with constant weight is no longer tenable. The author therefore reopens the question. He first secures a well-known formula. The limitations of mechanical possibilities call for further study, which leads to a simple formula for velocity that takes account both of the radial distance from the earth's center at the point of departure and at the point marking the end of burning, with incidental recognition of the variation of gravitational attraction with altitude. Air resistance is ignored throughout.   A. A. Bennett (Providence, R. I.).

Hemp, W. S. Note on the dynamics of a slightly deformable body. Coll. Aeronaut. Cranfield. Rep. no. 5, 5 pp. (1947).

The author gives an analysis of the motion of a slightly deformable body into three parts, namely: (1) motion of the center of mass, (2) rotation about the center of mass, which is defined by suitably specified moving axes, (3) motion relative to these axes, termed vibration. It is obvious from a known dynamical theorem that the first of these is independent of the other two, but it turns out that the last two are in general independent of each other only if both are small.   D. C. Lewis (Baltimore, Md.).

Kasner, Edward, and Mittleman, Don. On the initial curvature of dynamical trajectories. Univ. Nac. Tucumán. Revista A. 6, 71–79 (1947).

This paper deals with some properties of the motion of a particle moving in a plane in accordance with the differential equations $d^2x/dt^2 = \varphi(x, y)$, $d^2y/dt^2 = \psi(x, y)$. (This is called motion in an acceleration field of order r.) The following theorem is proved. In an acceleration field of order r, at a point at which the line of force has nth order contact with

its tangent line, the trajectory obtained by starting a particle from maximum rest (i.e., with $dx/dt, \cdots, d^{r-1}x/dt^{r-1}$, $dy/dt, \cdots, d^{r-1}y/dt^{r-1}$ all initially zero) also has contact of order n with the tangent line; and the limiting ratio of the departure of the trajectory to the departure of the line of force from the common tangent line is $1/\binom{n+r-1}{r-1}$. It is also proved that, if the particle starts at the instant $t=0$ under the above conditions, the coordinates x and y are, for small values of $|t|$, functions of t^r. *L. A. MacColl.*

Butenin, N. V. A survey of "degenerate" dynamical systems with the aid of a hypothesis of "jumping." Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 3–22 (1948). (Russian)

Dans ses leçons sur le frottement Painlevé montre que dans certains cas les lois de frottement de Coulomb appliquées aux corps solides conduisent à des paradoxes. L'auteur cite un exemple de F. Klein [Z. Math. Physik 58, 186–191 (1909)] illustrant les idées de Painlevé. Une étude approfondie de cette question a été faite [Klein, ibid.; von Mises, ibid., 191–195 (1909); Hamel, ibid., 195–196 (1909); Prandtl, ibid., 196–197 (1909)]. A la base se trouvent quatre hypothèses: (1) les corps sont absolument solides; (2) la pression normale entre les deux corps solides ne peut jamais être négative; (3) toutes les accélérations et les efforts sont finis, ainsi que les forces extérieures; (4) les lois de Coulomb sont valables. Pour éviter les paradoxes, Prandtl propose de remplacer l'hypothèse (3) par l'hypothèse d'un "saut."

Une étude théorique et une application pratique de l'hypothèse du "saut" a été faite par F. Pfeifer [Z. Math. Physik 58, 273–311 (1910)]. L'auteur expose d'une manière plus simple l'exemple de Pfeifer en construisant les trajectoires de la phase, ce qui permet d'étudier facilement les différents cas qui peuvent se présenter. Pour mieux illustrer l'hypothèse du "saut," l'auteur étudie deux exemples en utilisant la méthode du système dégénéré, c'est-à-dire des systèmes dans lesquels on supprime dans les équations différentielles du mouvement certains termes. Dans le premier exemple, un point mobile de masse m se meut sur une bande rugueuse animée d'un mouvement uniforme v_0. Le point peut rencontrer un obstacle solide. Le second exemple s'occupe de la théorie mécanique des oscillations de relaxation. *M. Kiveliovitch* (Paris).

Vâlcovici, Victor. Sur le principe de d'Alembert-Lagrange dans le cas des liaisons avec frottement de glissement. Acad. Roum. Bull. Sect. Sci. 26, 440–452 (1946).

L'auteur se propose de trouver la forme que le principe de d'Alembert-Lagrange prend au cas du frottement de glissement sur des surfaces (ou sur des courbes) mobiles et même déformables et il arrive à une forme générale des équations du mouvement qui comprend tous les cas des liaisons holonomes avec ou sans frottement. "On pourrait dire que de cette manière les forces de frottement de glissement entrent dans les chapitres de la mécanique analytique." Pour le cas le plus simple du mouvement d'un point de masse m sur une surface nondéformable on a : $-m\ddot{r}+F+\lambda\nabla\varphi-f|\lambda\nabla\varphi|\tau=0$, r étant le rayon vecteur, F la force appliquée, $\varphi(r, t)=0$ l'équation de la surface mobile, f le coefficient de frottement, τ le vecteur de la vitesse relative et λ une fonction inconnue de t. L'auteur considère le mouvement sur une courbe et le cas de n points mobiles. *O. Bottema* (Delft).

Vâlcovici, Victor. Le principe du "torseur." Un schéma vectoriel de la mécanique newtonienne. Acad. Roum. Bull. Sect. Sci. 27, 599–611 (1947).

Après un rappel de propriétés des torseurs, l'auteur, comme base de la mécanique, au lieu d'écrire que, en chacun des points d'un système de n points matériels, la force totale appliquée vaut la dérivée de l'impulsion mv, exprime, ce qui est équivalent, que pour tout groupe choisi de points matériels individualisés, le torseur des forces appliquées vaut la dérivée du torseur des impulsions. *M. Brelot* (Grenoble).

Vâlcovici, V. Sur certaines intégrales premières des équations du mouvement des systèmes. Acad. Roum. Bull. Sect. Sci. 25, 61–75 (1943).

The relations known to be satisfied by the momentum, angular momentum and kinetic energy of a dynamical system are written in compact form. A necessary and sufficient condition for the existence of certain kinds of first integrals is then seen to be the condition that a certain Pfaffian admit an integrating factor. The results are generalizations of Kotelnikoff's results [C. R. Acad. Sci. Paris 118, 129–131 (1894)]. A number of examples are considered. *D. C. Lewis* (Baltimore, Md.).

Castoldi, Luigi. I "movimenti astratti" di Appell e un nuovo esempio di vincoli anolonomi non lineari nelle velocità. Boll. Un. Mat. Ital. (3) 2, 221–228 (1947).

The author describes an explicit physical example of a nonholonomic system with a constraint equation nonlinear in the velocities; this system avoids a certain kind of limiting process usually involved in setting up such examples. It is, however, of course impossible to avoid all limiting processes. *D. C. Lewis* (Baltimore, Md.).

Signorini, Antonio. La meccanica razionale e la fisica matematica nell'Italia centrale e meridionale dal 1939 a oggi. Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 15, 17 pp. (1946).

Somigliana, C., Finzi, B., e Cattaneo, C. La meccanica razionale e la fisica matematica nell'Italia settentrionale e in Svizzera dal 1939 al 1945. Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 18, 21 pp. (1947).

Hydrodynamics, Aerodynamics, Acoustics

Jacob, Caïus. Sur l'extension de certaines formules intégrales aux écoulements des fluides parfaits. C. R. Acad. Sci. Paris 226, 1793–1795 (1948).

Levi-Civita [Atti Ist. Veneto Sci. 64, 1465–1472 (1905)] generalized a formula of Blasius, useful in calculating the force on a body in incompressible irrotational flow, for the three-dimensional case, as follows:

$$(1) \qquad \tfrac{1}{2}\int_S v^2 n d\sigma = \int_S (\mathbf{n}\cdot\mathbf{v})\mathbf{v} d\sigma,$$

where $\mathbf{v}$ is the fluid velocity vector, S is a closed surface including no singularities and $\mathbf{n}$ is the unit inward normal to S. The second Blasius formula was generalized by von Mises [Bull. Amer. Math. Soc. 50, 599–611 (1944); these

Rev. 6, 76], who obtained

$$(2) \qquad \tfrac{1}{2}\int_S v^2(\mathbf{r}\times\mathbf{n})d\sigma = \int_S (\mathbf{n}\cdot\mathbf{v})(\mathbf{r}\times\mathbf{v})d\sigma,$$

where $\mathbf{r}$ is the position vector. It is shown that (2) can be obtained directly from Green's formula. Another extension by von Mises, of a formula of Cauchy, is also discussed.

It is shown that there are formulas analogous to (1) and (2) for the steady irrotational flow of a barotropic fluid:

$$\tfrac{1}{2}\int_S K(v^2)\mathbf{n}d\sigma = \int_S \rho(\mathbf{n}\cdot\mathbf{v})\mathbf{v}d\sigma,$$

$$\tfrac{1}{2}\int_S K(v^2)(\mathbf{r}\times\mathbf{n})d\sigma = \int_S \rho(\mathbf{n}\cdot\mathbf{v})(\mathbf{r}\times\mathbf{v})d\sigma,$$

where ρ is the density and $-K(v^2)$ denotes $-\int\rho dv^2$, which is, by the equation of motion, the pressure.

W. R. Sears (Ithaca, N. Y.).

Jacob, Caïus. Sur un problème de M. Slioskine. Acad. Roum. Bull. Sect. Sci. 23, 263–265 (1942).

In a treatment of two-dimensional motion of a compressible fluid about a circular cylinder, for the case $\gamma=-1$, Slioskine [C. R. (Doklady) Acad. Sci. URSS (N.S.) 12 (1936 III), 419–421 (1936)] proposed two methods: either to establish a function $\Omega(u)$, u being a complex variable in the plane of the cylinder in an incompressible flow, and then to calculate the distorted contour from the transformation equation; or to find the function $\Omega(u)$ from a specified closed curve. In the latter procedure, the problem was reduced to solving a nonlinear integral equation which can be solved by successive approximations in powers of a parameter λ, of which the radius of convergence was given.

The author in this note continues Slioskine's problem to show that, if $\Omega(u)$ is analytic and regular everywhere outside the circle and the normal derivative of the real part T of $\Omega(u)$ on the circle, namely $u=e^{i\sigma}$, satisfies

$$\partial T/\partial n = 1 - \lambda\{ae^{-T(\sigma)} + 4b\sin^2\sigma e^{T(\sigma)}\},$$

where $\lambda>0$, $a+b=1$ and $1\leqq a\leqq\tfrac{1}{2}(1+\sqrt2)$, then the solution of the integral equation is unique if $\lambda>0$. Besides, λ should be determined in such a manner that T vanishes at infinity. This requires, on the circle, $F(\lambda)=\int_0^{2\pi}T(\sigma)d\sigma=0$.

Y. H. Kuo (Ithaca, N. Y.).

Castoldi, L. Sopra una proprietà dei moti permanenti di fluidi incomprimibili in cui le linee di corrente formano una congruenza normale de linee isotache. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 333–337 (1947).

The author proves two theorems. (1) In the steady motion of an incompressible fluid in which the streamlines are orthogonal trajectories of a singly infinite system of surfaces Σ, a necessary and sufficient condition that the streamlines are also lines of constant speed is that Σ is a family of surfaces of minimum area. (2) A necessary and sufficient condition that the motion described in (1) shall be realisable under conservative forces is that the surfaces Σ shall cut the streamlines proportionally. Example. The fluid rotates as a rigid body with angular velocity ω about an axis a. The streamlines are circles with a as axis; Σ is a family of planes through the axis a; the appropriate force potential is $-\omega^2r^2$.　　L. M. Milne-Thomson (Greenwich).

Vogelpohl, Georg. Die Strömung aus einer Wirbelquelle zwischen ebenen Wänden mit Berücksichtigung der Wandreibung. Z. Angew. Math. Mech. 24, 289–293 (1944).

The author considers a steady axially symmetric flow generated by a vortex-source combination between two parallel walls perpendicular to the axis of the vortex. The velocity components in the directions (r, θ, z) are designated respectively by (c_r, c_u, c_z). The mathematical problem has been simplified by assuming that (1) the motion is confined to planes, namely $c_z=0$, and (2) the velocity component c_u differs only slightly from that in the case of a perfect fluid, so that the dominant term in the viscous stress reduces to $\partial^2 c_u/\partial z^2$ alone. The Navier-Stokes equations then admit an exact solution with both c_r and c_u, aside from simple factors, expressed in terms of Weierstrass $\wp$-functions. It appears to the reviewer that the approximation holds only for small and moderate distances from the origin.　　Y. H. Kuo.

Schiller, Ludwig. Zur Herleitung der Ähnlichkeitsbedingungen aus der Identität der Differentialgleichungen. Z. Angew. Math. Mech. 24, 280–283 (1944).

The author shows that by introducing dimensionless variables in the manner proposed the differential equations can be made identical for all cases. The only remaining condition for similarity in that event is the identity of boundary conditions. As examples, the equation of harmonic oscillation and the Eulerian and Navier-Stokes equations are discussed.　　Y. H. Kuo (Ithaca, N. Y.).

Dolidze, D. E. A nonlinear boundary problem of the unstable motion of a viscous incompressible fluid. Akad. Nauk SSSR. Prikl. Mat. Meh. 12, 165–180 (1948). (Russian)

Soit, dans l'espace à trois dimensions (à noter: l'auteur étend ses conclusions au cas du plan) un domaine D (que nous supposerons borné, pour simplifier, bien que l'auteur traite complètement le cas extérieur) simplement connexe, limité par une surface F assez régulière. L'auteur se propose de démontrer l'existence et l'unicité d'un système de solutions, régulières dans D, des équations de Navier non-linéarisées, régissant le mouvement non-permanent d'une masse de fluide visqueux incompressible, emplissant D, les conditions aux limites imposées aux composantes $v_i(x_1, x_2, x_3, t)$ $(i=1, 2, 3)$ de la vitesse étant les suivantes: (1) les v_i sont connues dans D; (2) les $v_i(x_1, x_2, x_3, t)$ sont connues à tout instant sur F. Dans un but de simplification, l'auteur suppose nulles les forces de masse; mais il note qu'il serait facile de s'affranchir de cette restriction.

Voici, en gros, le marche suivie par l'auteur. Il forme d'abord le tenseur de Green

$$H_{ik}=\delta_{ik}h(P, Q, t)+\partial\varphi_i(P, Q, t)/\partial x_k+w_{ik}(P, Q, t),$$

$\delta_{ik}=1$ si $i=k$, $\delta_{ik}=0$ si $i\neq k$, où P et Q sont les points de coordonnées respectives x_1, x_2, x_3; y_1, y_2, y_3; $r=PQ$; où $h=-\tfrac{1}{2}(\pi\nu t)^{-\frac{3}{2}}\exp(-r^2/4\nu t)$; φ_i, une solution convenable de $\Delta\varphi_i=-\partial h/\partial x_i$; où w_{ik} sont des solutions convenables de $\nu\Delta w_{ik}-\partial w_{ik}/\partial t=\partial q_i/\partial x_k$; $\sum_k\partial w_{ik}/\partial x_k=0$ qu'un travail antérieur de l'auteur [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 11, 237–250 (1947); these Rev. 9, 116] permet de considérer comme connues. Cela permet de former un système de 12 équations intégrales non-linéaires aux inconnues $v_i\partial v_i/\partial x_k$ dont l'auteur construit une solution à l'aide d'un processus d'approximations successives, convergent pour $9\rho A\alpha(\pi t)^{\frac{1}{2}}<1$ (où A est une constante ne

dépendant que des vitesses à la frontière, α est une constante dépendant de D) et dont l'auteur prouve l'unicité.

L'auteur a donc abordé le problème fondamental de l'hydrodynamique des fluides visqueux, dont il a construit une solution locale. Il serait intéressant de comparer en détail ses méthodes et ses conclusions avec celles d'Oseen [Neuere Methoden und Ergebnisse in der Hydrodynamik, Akademische Verlagsgesellschaft, Leipzig, 1927] que l'auteur omet de citer, de même qu'il ne fait pas état des résultats globaux de Leray [Acta Math. **63**, 193–248 (1934)].

J. Kravtchenko (Grenoble).

Kampé de Fériet, J. Harmonic analysis of the two-dimensional flow of an incompressible viscous fluid. Quart. Appl. Math. **6**, 1–13 (1948).

The author considers plane flows of an incompressible viscous fluid for which the velocity vanishes on the boundary. Let $\Psi(\omega_1, \omega_2, t) = \frac{1}{2}\pi^{-2}\int_D \psi(x, y, t)e^{-i(\omega_1 x + \omega_2 y)}dxdy$ be the Fourier transform of the stream function $\psi(x, y, t)$ (D being the flow region), and similarly $u(\omega_1, \omega_2, t), v(\omega_1, \omega_2, t), z(\omega_1, \omega_2, t)$ the transforms of the x and y components of velocity and of the vorticity $\zeta = -\frac{1}{2}\Delta\psi$, respectively; also let $\gamma(\omega_1, \omega_2, t)$ be the spectral function for the kinetic energy of the flow, i.e., $\gamma = 2\pi^2(|u|^2 + |v|^2)$, $E = \int \gamma(\omega_1, \omega_2, t)d\omega_1 d\omega_2$. Simple formulas then express Ψ, u, v, γ in terms of z. For example,

$$\Psi(\omega_1, \omega_2, t) = 2z(\omega_1, \omega_2, t)/(\omega_1^2 + \omega_2^2),$$
$$\gamma(\omega_1, \omega_2, t) = 8\pi^2|z(\omega_1, \omega_2, t)|^2/(\omega_1^2 + \omega_2^2).$$

The derivations depend only on the definitions of the various quantities and not on the equations of motion. By means of the author's inequality on the rate of decrease of the kinetic energy of the fluid, the following bounds are obtained:

$$|z(\omega_1, \omega_2, t)| \leq \frac{1}{2}(2sE_0)^{\frac{1}{2}}\pi^{-2}(|\omega_1| + |\omega_2|) \exp(-4\nu k^{-1}t),$$
$$\gamma(\omega_1, \omega_2, t) \leq \frac{1}{2}sE_0\pi^{-2} \exp(-8\nu k^{-1}t),$$

where E_0 is the initial value of E, s the area of D, k a constant depending only on D, and ν the kinematic viscosity. Finally, the author derives a nonlinear integrodifferential equation for the transform z which, he suggests, might be the starting point for the study of the time variation of the spectral function γ. *D. Gilbarg* (Bloomington, Ind.).

Bouligand, Georges. Sur un cas d'entraînement d'un liquide visqueux. C. R. Acad. Sci. Paris **226**, 1776–1778 (1948).

Lin, C. C. An introduction to the dynamics of compressible fluids. Tech. Rep. no. **F-TR-1166-ND** (**GDAM A-9-M I**). Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. iv+161 pp. (1947).

This set of lecture notes is essentially a course on the main types of problems considered in gas dynamics and their methods of solution. The emphasis is primarily on formal methods of solution of flow problems. Included is a discussion of the more or less standard material of gas dynamics, such as the one-dimensional theory of channel flows, the Janzen-Rayleigh method for subsonic flows, the Prandtl-Glauert linearized theory, the hodograph method and method of characteristics for two-dimensional flows, the shock wave relations and the shock polar, etc. This work differs from conventional treatments in its fuller presentation of the hodograph method and by a section on the variational method, which is usually ignored in texts on gas dynamics. There are, in addition, sections on wave propagation in a compressible medium and on flows with consideration of heat conduction and viscosity. It is advisable that the reader have some previous acquaintance with hydrodynamics, a first course being adequate preparation. *D. Gilbarg* (Bloomington, Ind.).

Stanyukovič, K. P. Certain exact solutions of the equations of gas dynamics for centrally symmetric motions. Doklady Akad. Nauk SSSR (N.S.) **60**, 1141–1144 (1948). (Russian)

Certain classes of solutions of the equations of unsteady nonisentropic one-dimensional gas flow are discussed. If u, p, ρ are the velocity, pressure and density functions, then these classes are essentially of the form $u = t^\alpha \xi(rt^\beta)$, $p\rho^{-\gamma} = t^{2\alpha}\sigma(rt^\beta)$, $\rho = t^\delta \eta(rt^\beta)$; here r and t are space and time coordinates; α, γ, δ are integration constants. These solutions are said to be of interest in astrophysical problems and in certain shock interaction problems. However, no specific problems are mentioned. *G. F. Carrier*.

Shapiro, Ascher H., and Edelman, Gilbert M. Tables for numerical solution of problems in compressible gas flow with energy effects. J. Appl. Mech. **15**, 169–175 (1948).

Schaefer, M. Remarks on the work: "Two boundary value problems in the theory of hyperbolic partial differential equations of the second order with applications to supersonic gas flow" by F. Frankl and R. Aleksyiva (Moscow). The Graduate Division of Applied Mathematics, Brown University. Translation **A9-T-4**, i+6 pp. (1948).

The paper appeared in Technische Hochschule Dresden, Peenemunde Archiv **44/2a** (1944). The paper by Frankl and Aleksyiva appeared in Rec. Math. [Mat. Sbornik] **41**, 483–502 (1934).

Nikolsky, A. A., and Taganoff, G. I. Flow of a gas in a local supersonic zone and some conditions for the breakdown of potential flow. The Graduate Division of Applied Mathematics, Brown University. Translation **A9-T-17**, i+43 pp. (1948).

The paper appeared in Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 481–502 (1946); these Rev. **8**, 237.

Frankl, F. I. Influence of the acceleration of slender bodies of rotational symmetry upon the resistance of the gas. The Graduate Division of Applied Mathematics, Brown University. Translation **A9-T-19**, i+11 pp. (1948).

The paper appeared in Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 521–524 (1946); these Rev. **8**, 238.

Frankl, F. I. Uniqueness of solution of the problem of supersonic flow past a wedge. The Graduate Division of Applied Mathematics, Brown University. Translation **A9-T-20**, i+9 pp. (1948).

The paper appeared in Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **10**, 421–424 (1946); these Rev. **8**, 415.

Sauer, R. Bemerkungen zur Prandtlschen Affintransformation für Strömungen mit Unterschallgeschwindigkeit. Z. Angew. Math. Mech. **24**, 277–279 (1944).

This is a study of the rules of change in the pressure coefficient at a point on the surface of a given body with the free stream Mach number under the assumption of

small perturbations. For general three-dimensional flows and for two-dimensional flows, the author obtains the well-known results. The author's rule for axially symmetric bodies is, however, only applicable to vanishingly small thickness ratios and is thus unnecessarily restricted [see W. R. Sears, Quart. Appl. Math. **5**, 89–91 (1947); these Rev. **8**, 540]. *H. S. Tsien* (Cambridge, Mass.).

Mangler, W. Zusammenhang zwischen ebenen und rotationssymmetrischen Grenzschichten in kompressiblen Flüssigkeiten. Z. Angew. Math. Mech. **28**, 97–103 (1948). (German. Russian summary)

The author gives a general transformation rule which relates a boundary layer problem for an axially symmetric case to a corresponding two-dimensional problem. Let s be the length of arc along the body and let n denote the length of the normal on s from a point; $r = r_0(s)$ is the equation giving the body contour. Define $\bar{s} = \int_0^s (r_0^2(s)/L^2) ds$, $\bar{n} = r_0(s) n/L$, $\psi(\bar{s}, \bar{n}) = L^{-1}\psi(s, n)$ (stream function), $\bar{p}(\bar{s}) = p(s)$ (pressure at the surface), $\bar{i}(\bar{s}, \bar{n}) = i(s, n)$ (enthalpy), $\bar{\rho}(\bar{s}, \bar{n}) = \rho(s, n)$ (density) and $\mu(\bar{s}, \bar{n}) = \mu(s, n)$ (viscosity). The boundary layer equations for the axially symmetric flow are then shown to be identical with the equations for two-dimensional flow.

Hence the solution of an axially symmetric case for a given pressure distribution $p(s)$ and contour $r_0(s)$ is reduced to the solution of a two-dimensional problem with $\bar{p}(\bar{s}) = p(s)$ and $\bar{s} = \int_0^s (r_0^2(s)/L^2) ds$. The important relation between the wall shearing stress in both cases comes out to be $\bar{\tau}_0^* = \tau_0^* \{ \int_0^s r_0^2(s) ds / s r_0^2(s) \}^{\frac{1}{2}}$, where $\tau_0^* = (\tau_0/\rho_a V^2)(s V/\nu_a)^{\frac{1}{2}}$, and where ρ_a and ν_a denote the values of density and kinematic viscosity in the free stream of velocity V. Several applications are given. *H. W. Liepmann.*

Meksyn, D. The laminar boundary-layer equations. I. Motion of an elliptic and circular cylinders. Proc. Roy. Soc. London. Ser. A. **192**, 545–567 (1948).

The author summarizes his results as follows. (a) The problem is reduced, for the front part of a cylinder, to a nonlinear differential equation of the same form as that studied by Falkner and Skan [Aeronaut. Res. Committee [London], Rep. and Memoranda, no. **1314** (1930)], namely, $f''' + ff'' = \lambda(1 - f'^2)$. This equation has been numerically solved by Hartree [Proc. Cambridge Philos. Soc. **33**, 223–239 (1937)], so that the evaluation of velocities and surface friction requires only very simple and short computations. Near the separation point the problem leads to a more complicated differential equation.

(b) Elliptic cylinder. The calculated and observed velocities agree up to and including the point $x = 1.457$, where x is the distance along the boundary of the ellipse from the forward stagnation point, expressed as a multiple of the minor axis. An approximate integration gives the separation point at about 108° 30′ from the stagnation point, as against 103° observed, where the angle corresponds to the usual elliptic coordinate. (c) Circular cylinder. In all cases the calculated surface friction shows good agreement with the observed results, and the results previously obtained from experimental pressure distributions, for the front part of the cylinder. In the case of the cylinder of diameter $d = 5.89$ inches the calculated results show good agreement with observations almost up to separation point. (d) Pressure consists of two terms: one is equal to the pressure of the potential flow, the other one depends on the thickness of the boundary layer. *H. W. Liepmann* (Pasadena, Calif.).

Panichkin, I. A. Forces acting on an oscillating airfoil in a supersonic gas flow. The Graduate Division of Applied Mathematics, Brown University. Translation A9-T-21 i+11 pp. (1948).

The paper appeared in Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **11**, 165–170 (1947); these Rev. **9**, 545.

Riegels, F. Profile mit vorgegebener Druckverteilung. Z. Angew. Math. Mech. **24**, 273–276 (1944).

The problem indicated by the title is handled within the scope of the thin-airfoil theory; i.e., a vortex-sheet and a source distribution are determined to produce approximately the desired velocity distribution. *W. R. Sears.*

Hantzsche, W., und Wendt, H. Zur Berechnung der Unterschallströmung um ein beliebiges dünnes Profil. Z. Angew. Math. Mech. **24**, 234–239 (1944).

In a previous paper [same Z. **22**, 72–86 (1942); these Rev. **4**, 177] the authors calculated the subsonic flow past certain airfoil profiles by the method of iteration in powers of the thickness ratio. In the present paper they undertake to extend the method to arbitrary profiles. They find that this can be done provided that the transformation can be found that transforms the region outside the profile in the ζ-plane to the region outside a circle. The ζ-plane is the plane of the distorted profile obtained by the affine transformation given by the Prandtl-Glauert rule.
W. R. Sears (Ithaca, N. Y.).

Karp, S. N., Shu, S. S., and Weil, H. Aerodynamics of the oscillating airfoil in compressible flow. Prepared under the supervision of M. A. Biot. Tech. Rep. no. F-TR-1167-ND (GDAM A-9-M III). Headquarters Air Materiel Command, Wright Field, Dayton, Ohio. v+57 pp. (1947).

This monograph constitutes a review of work on the subject, especially in Germany during 1939–1945. The investigations reported were all confined to thin airfoils of infinite span, treated by means of the linearized potential-flow theory, for both subsonic and supersonic speeds. Various methods of calculation are explained. A bibliography of 45 papers is appended. A detailed account of published tables of functions relating to this subject is given in the report reviewed below. *W. R. Sears.*

Karp, S. N., and Weil, H. The oscillating airfoil in compressible flow. II. A review of graphical and numerical data. Prepared under the supervision of M. A. Biot. Tech. Rep. no. F-TR-1195-ND (GDAM A-9-M III/II). Headquarters Air Materiel Command, Wright-Patterson Air Force Base, Dayton, Ohio. iii+44 pp. (1948). For part I see the preceding review.

Barton, M. V. Stability of an oscillating airfoil in supersonic airflow. J. Aeronaut. Sci. **15**, 371–376 (1948).

Cohen, Clarence B., and Evvard, John C. Graphical method of obtaining theoretical lift distributions on thin wings at supersonic speeds. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1676**, 42 pp. (1948).

Margolis, Kenneth. Supersonic wave drag of nonlifting sweptback tapered wings with Mach lines behind the line of maximum thickness. Tech. Notes Nat. Adv. Comm. Aeronaut., no. **1672**, 34 pp. (1948).

Ferrari, C. **The supersonic flow about a sharp nosed body of revolution.** The Graduate Division of Applied Mathematics, Brown University. Translation A9-T-18, i+22 pp. (1948).

The paper appeared in Aerotecnica 16, 121–130 (1936).

Pistolesi, Enrico, e Ferrari, Carlo. **L'aerodinamica in Italia (dal 1939 al 1945).** Pont. Acad. Sci. Relationes Auctis Sci. Temp. Belli 13, 47 pp. (1946).

Siestrunck, Raymond, et Fabri, Jean. **Sur l'équation générale des potentiels hélicoïdaux en fluide parfait compressible.** C. R. Acad. Sci. Paris 226, 1430–1431 (1948).

The authors state that the linearized equation for the velocity potential φ in cylindrical coordinates (z, θ, r),

$$\varphi_{zz}+\varphi_{rr}+r^{-1}\varphi_r+r^{-2}\varphi_{\theta\theta}=a_0^{-2}\varphi_{tt},$$

when transformed into coordinates fixed with respect to an observer moving in a helicoidal path, is the Goldstein equation $\varphi_{rr}+r^{-1}\varphi_r+(r^{-2}+R^{-2}\lambda^{-2})\varphi_{\zeta\zeta}=0$, where $\zeta=\theta-z/\lambda$ and λ is the pitch of the helix. Therefore, as far as the first-order effect is concerned, the induced velocity at the propeller blade due to the helicoidal vortex of given strength generated by the propeller is the same as in incompressible flow. A similar statement for the wing is shown by Tsien and Lees [J. Aeronaut. Sci. 12, 173–187, 202 (1945); these Rev. 6, 249]. The calculation by Tsien and Lees on the induced velocity of the helicoidal vortex generated by the propeller is, however, not correct. *H. S. Tsien.*

Wieghardt, K. **Über die turbulente Strömung im Rohr und längs einer Platte.** Z. Angew. Math. Mech. 24, 294–296 (1944).

Bei der rechnerischen Übertragung der turbulenten Strömung in einem Rohr auf die längs einer ebenen Platte ergeben sich gewisse Abweichungen, die durch den verschiedenen Turbulenzzustand in Rohrmitte und am Rand der Plattenreibungsschicht erklärt werden.

Author's summary.

Görtler, H. **Einige Bemerkungen über Strömungen in rotierenden Flüssigkeiten.** Z. Angew. Math. Mech. 24, 210–214 (1944).

Reference is made to the theoretical and experimental work of J. Proudman and of G. I. Taylor [see Taylor, Proc. Roy. Soc. London. Ser. A. 104, 213–218 (1923) and previous papers mentioned there] in which it was shown that the steady motion produced by the slow movement of any submerged body relative to a rotating fluid is two-dimensional; i.e., a cylinder of fluid, with generators parallel to the axis of rotation, moves with the body. The author considers slow oscillatory motions in an inviscid fluid rotating with angular velocity ω. The differential equations of this system are of elliptic type for $\beta > |2\omega|$ and hyperbolic for $\beta < |2\omega|$, where β is the circular frequency of the oscillation. In the latter case, the characteristic surfaces are cones emanating from a point of disturbance; in Taylor's case $\beta \to 0$ and the cones become the generators of the cylinder mentioned above. An analogy is drawn between the significance of the critical frequency $\beta = |2\omega|$ and the analogous critical frequencies in meteorology. Proceeding to the case of plane incompressible inviscid flow in a fluid rotating between two plates, the author draws some general conclusions regarding the similarities of such flows and corre-

sponding flows in a nonrotating fluid, and the creation of vortices in such cases. *W. R. Sears* (Ithaca, N. Y.).

Burgers, J. M. **On the influence of gravity upon the expansion of a gas.** I. Nederl. Akad. Wetensch., Proc. 51, 145–154 (1948). (English. Esperanto summary)

The problem is that of the expansion of a semi-infinite vertical column of perfect gas into a vacuum with the upper partition suddenly removed at the instant $t=0$. The gas column is kept in isentropic equilibrium against a constant gravitational force for $t<0$. The ratio of specific heats is taken as 5/3. The problem is solved by Riemann's characteristic method. It is shown that the flow develops shocks and then the solution breaks down because of the change in entropy. Aside from this uncertainty, the solution obtained shows that the gas settles down to a new equilibrium state and that the main body of gas moves upward only slightly between the initial and the final equilibrium states.

H. S. Tsien (Cambridge, Mass.).

Pack, D. C. **On the formation of shock-waves in supersonic gas jets. (Two-dimensional flow.)** Quart. J. Mech. Appl. Math. 1, 1–17 (1948).

When Mach waves form an envelope, a shock wave is developed. This fact is utilized by the author to determine the location and shape of the shock formed in a two-dimensional jet issuing from a nozzle at supersonic speed and at a pressure higher than the surrounding medium. The calculations are carried out by using the characteristic method. The author shows that, at moderate exit pressure, the shock is developed from compression Mach waves reflected from the jet boundary after crossing the axis of the jet. In this case, the shock is inclined toward the downstream direction. At higher exit pressure, the shock is developed from the compression Mach waves before crossing the axis of the jet. Then the shock is inclined forward and the starting point of the shock is closer to the nozzle. At very high exit pressure and sonic exit velocity, approaching conditions at the muzzle of a gun, the shock starts very close to the muzzle at a large angle from the plane of the muzzle.

The author neglects the vorticity in the flow after the shock on the basis of the small curvature of the shock. However, it should be noted that the vorticity is not only a function of the shock curvature in this case, since the flow before the shock is not uniform. *H. S. Tsien.*

Kai, Fritz. **Ein Beitrag zur Theorie der Wellen in freien Gasstrahlen.** Z. Angew. Math. Mech. 28, 80–85 (1948).

This is an independent investigation, with the same result, of the problem studied by Pack [see the preceding review]. The author also neglects the vorticity generated by the shock. The procedure for determining the shock location seems, however, more straightforward than Pack's. It is likely that unless the exit pressure is only infinitesimally higher than the surrounding medium, the wave pattern never really repeats itself and the jet is damped by the successive formation of shocks. *H. S. Tsien.*

Wuest, Walter. **Zur Theorie des gegabelten Verdichtungsstosses.** Z. Angew. Math. Mech. 28, 73–80 (1948). (German. Russian summary)

By using curves which express the flow-deflection angle as a function of the shock-wave angle for various values of the free-stream Mach number, a simple graphical method is devised for the construction of bifurcate shock configura-

tions. The range of Mach numbers is obtained for which such shocks can exist.   *E. N. Nilson.*

Schultz-Grunow, F. Zur Behandlung nichtstationärer Verdichtungsstösse und Detonationswellen. Z. Angew. Math. Mech. **24,** 284–288 (1944).

Es wird gezeigt, dass man die Veränderlichkeit eines Verdichtungsstosses und auch einer Detonationswelle in einem Zustandsdiagramm, dessen Koordinaten die Gas- und die Schallgeschwindigkeit sind, besonders leicht verfolgen kann.   *Author's summary.*

Starr, Victor P. Momentum and energy integrals for gravity waves of finite height. J. Marine Research **6,** 175–193 (1947).

The author is concerned with gravity waves of finite amplitude in a channel of constant or of infinite depth. Without making use of exact solutions he derives by means of various plausible assumptions a number of relations for periodic waves and for solitary waves. Some of these relations connect the kinetic energies of horizontal and vertical motions with the potential energy of the wave. Others involve the speed of propagation of the wave and the depth of the liquid.   *F. John* (New York, N. Y.).

Oswatitsch, Kl. Die Verdunstungsgeschwindigkeit von Wolken. (Die numerische Integration eines Wärmeleitungsvorganges.) Z. Angew. Math. Mech. **24,** 257–263 (1944).

A system of three nonlinear partial differential equations of the second order is set up, derived from the equations of molecular diffusion and heat conduction, and a new equation for the speed of evaporation of individual droplets. The equations define the distribution of temperature, vapor pressure and volume of the fog or cloud as function of time and a single space coordinate. An iterative process for the integration of the equations is suggested and illustrated by an example. The results show that the effects of molecular diffusion and conduction are unimportant in the spreading and dissolution of large scale fogs and clouds.   *H. Panofsky* (New York, N. Y.).

Jaw, Jeou-jang. Theory of unstationary wind-current. Sci. Rep. Nat. Tsing Hua Univ. **4,** 363–378 (1947).

The problem of drift currents produced by unstationary winds is solved by an operational method, the coefficient of eddy viscosity being treated as a constant. The results are applied to three cases: a wind fluctuating periodically, a wind suddenly stopping and a wind changing in direction. All results indicate large systematic deviations from Ekman's stationary wind solution.   *H. Panofsky.*

Gião, António. Solution générale du problème de la prévision mathématique du temps à échéance quelconque. Soc. Geograf. Lisboa. Bol. **60,** 233–272 (1942).

The aim of the paper is to forecast values of a meteorological variable, given a suitable set of initial values. First conditions are found for the wave equation to possess a solution ψ inside and on a surface S, such that $\partial\psi/\partial n = 0$, $\psi = u$ on S, u being a given function of position and time. Next a rather obscure argument is expounded, enabling the values of u on S at an arbitrary time to be constructed from those of ψ inside S during a suitable initial interval. Finally, by an argument which the reviewer can neither understand nor credit, any general meteorological variable is identified with the solution ψ of the wave equation, so that the prediction problem is solved.   *T. G. Cowling.*

Gião, António. Nouvelles recherches sur les perturbations spontanées du mouvement des fluides avec des applications à l'hydrodynamique solaire. Soc. Geograf. Lisboa. Bol. **61,** 509–522 (1943); **62,** 35–94, 201–256 (1944).

The motion in a fluid medium under steady external actions is divided into the steady ("entretenu") part which those actions could maintain, and a perturbation part. The first half of the paper derives the equations of these two parts in recognizably usual forms; the derivation starts from the equations of a finite mass, and proceeds to those valid at a point. The entretenu motion is supposed assigned; viscous stresses are taken as sole cause of the difference between the entretenu and actual motions, and are replaced in the equations by terms representing the entretenu motion.

In solar hydrodynamics, the sun's steady rotation is taken as the entretenu motion. It is determined, with good agreement with observation, by taking the body force of friction to vanish, with a supplementary assumption: surface viscous stresses are not considered. Sunspots are taken as the perturbations. They are regarded purely as vortices; their thermal properties are not considered, and their magnetic properties are interpreted simply as implying a related vorticity. Suggested explanations of their stability, distribution with latitude, and vorticity and other properties are advanced.   *T. G. Cowling* (Leeds).

Gião, António. Le problème atmosphérique d'après la théorie des perturbations spontanées. Portugaliae Phys. **2,** 203–234 (1947).

The equations derived in the paper reviewed above are applied to meteorology, after simplifying by omitting certain terms. Two vectors, defined as complicated functions of the "entretenu" motion, are used to calculate the motion of fronts. A differential equation is found for the pressure at the earth's surface in terms of the same two vectors, and is solved formally by successive approximations and by expansion in terms of normal oscillations. A method of solution of the general equations by successive approximations is also indicated.   *T. G. Cowling* (Leeds).

Florin, V. A. The basic equation of the consolidation of an earth mass. Doklady Akad. Nauk SSSR (N.S.) **59,** 21–24 (1948). (Russian)

Florin, V. A. The problem of consolidation of an earth mass. Doklady Akad. Nauk SSSR (N.S.) **59,** 219–222 (1948). (Russian)

The second paper is a continuation of the first. A three phase earth mass consisting of a liquid, a solid and a gaseous phase is considered, its voids ratio ϵ being a function of the sum of the principal stresses θ. The relative proportion of the three phases $(m+n+s=1)$ is assumed to be the same so far as any part of the volume of the mass and the area of any cross section is concerned. This relative proportion is easily connected with the average moisture content η of the mass. The author designates by u, v and w what he calls the "seepage velocities" of each of the three phases, respectively, and writes the equations of the discharge of each of the three phases beyond the boundaries of the earth mass. He combines the three equations in question considering the density of the gas ρ as a function of the gas pressure $p+p_0$, the latter symbol standing for the atmospheric pressure, after which the following equation is obtained:

$$\operatorname{div}(u+v+w) = \frac{1}{p+p_0}(s+\mu m)\cdot\frac{\partial p}{\partial t} + \frac{1}{\rho}(w,\operatorname{grad}\rho) = 0,$$

in which μ is the coefficient of solubility of the gaseous phase in the liquid, t being time.

Further work is done using the Davey formula for the velocity of the liquid phase with respect to the soil skeleton (solid phase). Neglecting small values, the author obtains the basic equation of consolidation in the form

$$\frac{1}{1+\epsilon}\cdot\frac{\partial\epsilon}{\partial t}+\beta\cdot\frac{\partial p}{\partial t}-\text{div }\kappa\cdot\text{grad }H=0,$$

where κ is the average coefficient of permeability of the mass, H the hydraulic head and β a value to be determined experimentally. In the second paper the variability of the values κ and θ is considered. Two sketches illustrate the numerical solution of a corresponding problem.

D. P. Krynine (New Haven, Conn.).

Keller, Joseph B., and Keller, Herbert B. **Reflection and transmission of sound by a spherical shell.** J. Acoust. Soc. Amer. 20, 310–313 (1948).

A periodic point source is at the center of a sphere. This sphere is surrounded by a second sphere of different density and radius. The authors solve the problem of finding the reflection and transmission of sound in these three media (i.e., inside the first sphere, the spherical shell and exterior to the second sphere). All of these regions have different radii and densities. A modification of a method due to Rayleigh [The Theory of Sound, 2d ed., Macmillan, London, 1896, v. 2, pp. 86–88] is used. A comparison is made with the work of H. Primakoff and J. B. Keller on thin curved shells [same J. 19, 820–832 (1947); these Rev. 9, 315].

A. E. Heins (Pittsburgh, Pa.).

Elasticity, Plasticity

Leĭbenzon, L. S. **Kurs Teorii Uprugosti.** [Course in the Theory of Elasticity]. 2d ed. OGIZ, Moscow-Leningrad, 1947. 464 pp.

The first edition, published in 1941 under the title "A Brief Course in the Theory of Elasticity," was designed to provide a solid basis for the study of specialized topics in elasticity. It contained no discussion of the theory of shells and the nonlinear theory of elasticity. The revised volume contains three new introductory chapters: Theory of bending of thin plates, Equations of plastic deformation and Variational methods of solution of elastic problems. The presentation is clear and concise, and the level of the book approximates that of R. V. Southwell's Introduction to the Theory of Elasticity [2d ed., Oxford, 1941]. In order to make the book accessible to a wider circle of readers the author avoids the use of vector and tensor calculus and follows the symbolism of Love's Treatise on the Mathematical Theory of Elasticity [4th ed., Cambridge University Press, 1927].

The first five chapters [139 pages] contain the analysis of deformed and stressed states and include the derivation of the equilibrium equations. A noteworthy feature is the formulation of the equilibrium equation in terms of Galerkin's and Neuber and Papkovitch's stress functions.

Chapters 6 and 7 [47 pages] deal with the classical 3-dimensional problems of elastic equilibrium of the sphere and the problems of Boussinesq and Hertz. Chapters 8, 9 and 10 are concerned with two-dimensional elastic problems:

plane stress and strain, and Saint Venant's theory of flexure and torsion. A brief account of the use of the complex variable theory in plane problems, based on the work of Kolossoff and Muschelišvili, is included. Chapter 11 [23 pages] deals with variational principles of elasticity and establishes the connection of Castigliano's principle with Saint Venant's compatibility conditions [first pointed out by Southwell in 1936]. The thermo-elastic equations are given in chapter 12 and the rudiments of the thin plate theory are presented in the space of less than thirty pages in chapter 13. The inclusion of a brief account of the theory of plasticity [chapter 14] in a book on elasticity is a welcome departure from the usual practice. This contains a sketch of the contributions to the theory by von Mises, Hencky, Prandtl, Reiss and Prager.

Chapter 15, on elastic waves [21 pages], gives a summary of basic facts about the propagation of elastic waves, and the concluding chapter 16 furnishes a formal introduction to direct methods of solution of variational problems in elasticity. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Aymerich, Giuseppe. **Configurazioni coniugate di sforzi nell'elasticità piana.** Rend. Sem. Fac. Sci. Univ. Cagliari 16 (1946), 145–148 (1948).

Neuber, H. **Vereinfachtes Verfahren zur Spannungsberechnung in dünnwandigen prismatischen Hohlkörpern unter Innendruck.** Z. Angew. Math. Mech. 28, 187–189 (1948).

Hill, R. **A variational principle of maximum plastic work in classical plasticity.** Quart. J. Mech. Appl. Math. 1, 18–28 (1948).

The paper is concerned with the following boundary value problem: if the velocities are given on the surface of a body which is made of a von Mises material and if it can be assumed that the whole body is in the plastic state, to find the stresses in the interior of the body. It is shown that the solution of this problem, if it exists at all, is unique. Moreover, of all distributions of stress which satisfy the equations of equilibrium and the yield condition, the actual one corresponds to the maximum power of deformation. This variational principle is applied to the stress distribution in a prismatic bar of arbitrary cross section which is plastically deformed by forces and couples applied to its ends. The relation between this variational principle and Sadowsky's principle of maximum effort [J. Appl. Mech. 10, A-65–A-68 (1943); these Rev. 4, 263] is discussed. *W. Prager.*

Wang, Chi-Teh. **Bending of rectangular plates with large deflections.** Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1462, 34 pp. (1948).

The system of nonlinear differential equations for the bending of rectangular plates with large deflections, due to von Kármán, is replaced by the equivalent finite difference equations. The difference equations are solved by the method of successive approximations for two cases of rectangular plates with ratios of length to width of 1.5 and 2. Tabular and graphical results are given. *D. L. Holl.*

Willers, Fr. A. **Das Falten des Randes beim Pressen von Schalen.** Z. Angew. Math. Mech. 24, 297–300 (1944).

Für die Anzahl der Wellen, die an dem schmalen Rande kreisrunder, zu Schalen gepresster Scheiben auftreten, wird mittels des Ritzschen Verfahrens eine Näherungsformel abgeleitet. *Author's summary.*

Conway, H. D. **The large deflection of simply supported beams.** Philos. Mag. (7) **38**, 905–911 (1947).

The author considers the deflection of an elastic beam, taking into account the finite slope of the beam and the fact that the support reactions are not vertical but rather consist of a normal reaction plus a frictional drag. The results are expressed in terms of elliptic functions.   *G. F. Carrier.*

Scanlan, R. H. **A note on transverse bending of beams having both translating and rotating mass elements.** J. Aeronaut. Sci. **15**, 425–426, 434 (1948).

Williamson, Robert A. **Torsion-bending stresses in box beams.** J. Aeronaut. Sci. **15**, 427–434 (1948).

Rahmatulin, H. A. **On the propagation of cylindrical waves in plastic deformations (torsional impact).** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 39–46 (1948). (Russian)

This paper is concerned with the following problem: a rigid, infinitely long cylinder is embedded in an infinite, plastic body with linear strain-hardening; given the time dependence of the exterior torque applied to this cylinder, to study the propagation of the shear waves produced in the plastic body.   *W. Prager* (Providence, R. I.).

Sokolovskiĭ, V. V. **The propagation of cylindrical shear waves in an elastic-viscoplastic medium.** Doklady Akad. Nauk SSSR (N.S.) **60**, 1325–1328 (1948). (Russian)

This discussion is based on a stress-strain law suggested by K. Hohenemser and W. Prager [Z. Angew. Math. Mech. **12**, 216–226 (1932)]. According to this law the material behaves in an elastic manner as long as the shearing stress does not exceed the yield limit; beyond this limit the shear rate is obtained as the sum of an elastic component which is proportional to the rate of the shearing stress, and a viscous component which is proportional to the excess of the shearing stress over the yield stress. Referred to cylindrical coordinates, the type of motion studied in the paper is characterized by the vanishing of the radial and axial velocity components; the circumferential velocity component is assumed to depend only on the radius r and the

time t. The equations of motion are established and their integration by the method of characteristics is discussed. An illustrative example is worked out in detail.
          W. Prager (Providence, R. I.).

Kromm, Alexander. **Zur Ausbreitung von Stosswellen in Kreislochscheiben.** Z. Angew. Math. Mech. **28**, 104–114 (1948).

The propagation of elastic waves in a plane with a circular hole is investigated subject to a sudden rotary-symmetric disturbance at the edge of the hole. The problem is handled with the aid of the Laplace transform. From this transformation, an integral equation of the Volterra type is derived. It is remarked that this integral equation can be easily solved numerically.   *A. E. Heins.*

Fridman, M. M. **The diffraction of a plane elastic wave by a semi-infinite rectilinear rigidly fastened slit.** Doklady Akad. Nauk SSSR (N.S.) **60**, 1145–1148 (1948). (Russian)

The note contains a solution of the two-dimensional problem of diffraction of a plane (longitudinal or transverse) elastic wave by a semi-infinite rigidly fixed rectilinear slit. The solution is obtained by making use of the functionally invariant solutions of the wave equation deduced by V. I. Smirnov and S. L. Sobolev, and by applying the methods of N. I. Mushelišvili's Singular Integral Equations [Moscow-Leningrad, 1946; these Rev. **8**, 586].
          I. S. Sokolnikoff (Los Angeles, Calif.).

Kufarev, P. P. **On a special case of the oscillation of a spiral spring with touching coils.** Akad. Nauk SSSR. Prikl. Mat. Meh. **12**, 209–210 (1948). (Russian)

The case treated is that of a spiral spring satisfying assumption (A): at each instant of time each coil is in contact with at most one of the two adjacent coils, either the coil just above or the one just below. In addition, a simplifying assumption is made concerning the force between two coils in contact. An example is worked out.
          J. B. Dias (Providence, R. I.).

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Herrenden-Harker, G. F. **Caustics by reflection in a concave spherical reflecting surface.** Amer. J. Phys. **16**, 272–284 (1948).

Bouyer, Roger. **Sur l'aberration primaire de sphéricité des lentilles minces.** Rev. Optique **27**, 288–294 (1948).

The author analyzes the first order spherical aberration of a thick lens under the assumption that the marginal thickness is negligible. The result is given as a sum of four terms and the geometrical significance of each of these terms is given.   *M. Herzberger* (Rochester, N. Y.).

Meyer-Eppler, W. **Die funktionalanalytische Behandlung des Schattenproblems.** Optik **1**, 465–474 (1946).

In 3-dimensional Cartesian coordinates, let L, S, A denote respectively the planes $z=0$, $z=a$, $z=a+b$. Let (x, y), (ξ, η), (u, v) be 2-dimensional Cartesian coordinate systems in these planes, connected with the (x, y, z)-coordinates by the relations $(x, y)=(x, y, 0)$; $(\xi, \eta)=(-\xi, -\eta, a)$;

$(u, v)=(-u, -v, a+b)$. Suppose that L is luminous, with luminosity-distribution function $F(x, y)$, and that S is transparent, with transparency-distribution function $T(\xi, \eta)$. Then, assuming that all rays lie near the Cartesian z-axis, the brightness distribution $E(u, v)$ on A is given by

$$E(u, v) = c \int\int_{-\infty}^{\infty} F(x, y) T\left(\frac{(p-1)u-x}{p}, \frac{(p-1)v-y}{p}\right) \frac{dxdy}{(a+b)^2},$$

where $p=(a+b)/b$ and c is a constant.

Integral relations of the type

$$E(u, v) = \int\int_{-\infty}^{\infty} F(x, y) T(u-x, v-y) dxdy$$

can thus be investigated photometrically. Among other particular applications, arrangements for periodogram analysis, for low-pass filtering and for integration are suggested.
          E. H. Linfoot (Cambridge, England).

Weinstein, W. The reflectivity and transmissivity of multiple thin coatings. J. Opt. Soc. Amer. 37, 576–581 (1947).

General expressions for the transmissivity and reflectivity of multilayer coatings of any number of coatings, for light polarized in any way and incident at any angle, are derived. Two methods of solution are developed, each giving the ratios between the incident and reflected amplitudes, and the incident and transmitted amplitudes. One method yields explicit formulae for the transmittance and reflectance. The other, described as more convenient for calculation, expresses the solutions in matrix notation.

A. J. Kavanagh (Buffalo, N. Y.).

Abelès, Florin. Transmission de la lumière à travers un système de lames minces alternées. C. R. Acad. Sci. Paris 226, 1808–1810 (1948).

Mihul, Constantin. Réflexion des ondes électromagnétiques par des milieux aux constantes optiques variables d'une façon continue. Disquisit. Math. Phys. 1, 253–305 (1940).

The author first obtains a formula for the reflection coefficient of a plane electromagnetic wave incident normally or obliquely at the surface of a stratified medium. He assumes that the dielectric constant ϵ and conductivity σ are constant in each layer, with abrupt discontinuities at each surface of separation between successive layers. For a large number of thin layers this should be an approximate solution for reflection from a medium in which the parameters vary continuously in a direction normal to the surface. Fresnel's reflection and transmission coefficients are used at each surface of separation, and it is shown that contributions to the reflected field due to multiple internal reflections may be neglected.

The theory is then applied to a discussion of reflections from the ionosphere, in which variations in ϵ and σ are due to variations in N (the number of electrons per cm³) and f (the collision frequency). The author distinguishes between two types of reflection which he calls vitreous and metallic; in the former, variations in the reflection coefficient are mainly due to variations in ϵ (high frequency case), in the latter to variations in σ (low frequency case). The reflective power of the ionosphere is evaluated in each case for various simple assumptions regarding the variation of f and N with height, and some agreement with recent experimental results obtained. *M. C. Gray* (New York, N. Y.).

Polara, V. Sulla birifrangenza nei cristalli. Matematiche, Catania 2, 41–64 (1946).

Fock, V. A. New methods in diffraction theory. Philos. Mag. (7) 39, 149–155 (1948).

This paper gives a general discussion of an approximate method of solution of the problem of the diffraction of electromagnetic waves by bodies of arbitrary shape based on "the principle of the local field in the penumbra region." This states that the diffraction of the incident field takes place in a narrow strip along the boundary of the geometrical shadow, and that the field in this strip depends only on the value of the incident wave in its immediate neighborhood, on the electrical properties of the body and its geometrical shape near the strip. It is assumed that the body is a good conductor so that the skin depth is small, and that the radius of curvature throughout the strip is large compared with the wave-length. *M. C. Gray* (New York, N. Y.).

Kahan, Théo, et Eckart, G. L'onde de surface de Sommerfeld. Solution définitive d'un problème resté depuis longtemps en suspens. C. R. Acad. Sci. Paris 226, 1513–1515 (1948).

The authors aim at showing that, in order to obtain the true solution of the problem of a radiating vertical electric dipole on the plane earth, the surface-wave term has to be omitted from Sommerfeld's classical solution [Ann. Physik (4) 28, 665–736 (1909)], thus arriving at the same conclusion as Epstein [Proc. Nat. Acad. Sci. U. S. A. 33, 195–199 (1947); these Rev. 9, 126], whose paper they try to correct with respect to a question of minor importance. The reviewer cannot agree with the theory of the authors. In addition to the critical remarks given before in the review of Epstein's cited paper, the reviewer wishes to state that (1) there is no mathematical disagreement between the basic expression of Sommerfeld on the one hand and that of Weyl [Ann. Physik (4) 60, 481–500 (1919)] on the other; (2) the authors seem to have overlooked the fact that Sommerfeld's transformation of his basic expression into the sum of a surface-wave term P and a space-wave term Q does not hold for real-valued wave-numbers k_1 and k_2, as was already pointed out by Sommerfeld himself; (3) it is mathematically immaterial whether or not some part of the complete solution, for instance P, violates certain boundary conditions, the behavior of $P+Q$ only being conclusive; (4) Q alone cannot serve as a solution since Q is logarithmically singular on the whole z-axis. *C. J. Bouwkamp.*

Bucerius, H. Theorie des Regenbogens und der Glorie. Optik 1, 188–212 (1946).

The problem of the diffraction of plane polarised light by a dielectric sphere can be readily solved by means of an infinite series of spherical wave functions. The resulting formula for the intensity of the scattered light is valid for all values of $x = 2\pi a/\lambda$, where a is the radius of the sphere, λ the wave length. When x is small this series can be used to compute the intensity, and the results were used by Mie in the theory of the absorption of light by colloidal gold solutions.

When x is large, the series is quite useless for computing the intensity. In the present paper, asymptotic formulae are derived for the case x large by a method which is, it is claimed, much simpler than that used by Debye in the corresponding problem of the dielectric cylinder. The method consists of using known approximations to the spherical wave functions, replacing a sum of a large number of terms by an integral, and then approximating by the method of stationary phase. The results are applied to the theory of the rainbow, and of the phenomenon that the shadow of an observer on a bank of fog often appears to be surrounded by coloured rings. *E. T. Copson* (Dundee).

Kubelka, Paul. New contributions to the optics of intensely light-scattering materials. I. J. Opt. Soc. Amer. 38, 448–457 (1948).

It is shown that the theory of Kubelka and Munk [Z. Techn. Phys. 12, 593–601 (1931)], originally derived on the assumption of completely diffused incident light, is also applicable to other conditions, for example, collimated light incident at 60° on a perfectly diffusing material. A number of new formulas for practical use are developed, and some of these are shown to coincide with the formulas of Gurevič [Phys. Z. 31, 753–763 (1930)] and of Judd [J. Research Nat. Bur. Standards 12, 345–351 (1934); 13, 281–292 (1934)]. *W. E. K. Middleton* (Ottawa, Ont.).

Lax, M., and Feshbach, H. Absorption and scattering for impedance boundary conditions on spheres and circular cylinders. J. Acoust. Soc. Amer. **20**, 108–124 (1948).

The paper presents formulas and tables for the calculation of scattering and absorption of plane waves by a spherical or a circularly cylindrical obstacle. The incident wave is expressed as a sum of modes each of which has a characteristic angular dependence. The effect of the obstacle exhibits itself at large distances by a phase shift η_n of the radial dependence associated with the nth type of angular dependence. These phase shifts are determined by the boundary conditions at the scattering surface, which in turn depend on the nature of the acoustic material applied, viz.,

$$\eta_n = \delta_n - 90^\circ + \tan^{-1}\{U_n - i(\gamma_n - i\sigma_n)T_n\},$$

where $\gamma_n - i\sigma_n$ is the specific admittance of the surface, associated with the nth type of angular dependence. The auxiliary functions $U_n(x)$ and $T_n(x)$ are defined by $T_n = x^2(j_n^2 + y_n^2)$ (spherical case), $T_n = \frac{1}{2}\pi x(J_n^2 + Y_n^2)$ (cylindrical case); $U_n = \tan(\delta_n' - \delta_n + 90^\circ)$, where the remaining symbols have been previously defined [A. N. Lowan, P. M. Morse, H. Feshbach, and M. Lax, Scattering and Radiation from Circular Cylinders and Spheres, Tables of Amplitudes and Phase Angles, U. S. Navy Department, Office of Research and Inventions, 1946; these Rev. **8**, 491]. A knowledge of the η_n permits the complete determination of the scattering and absorption as well as the angular distribution of the scattered radiation. By far the greater part of the paper is occupied by numerical values of $T_n(x)$ and $U_n(x)$. In these tables $n = 0(1)18$ or 19, $x = 0(0.1)10$, while six significant figures are given so far as the functions do not exceed 10^7. Tables concerning η_n will be published shortly.   *C. J. Bouwkamp* (Eindhoven).

Takeno, Hyôitirô. A generalization of Rumer's form of Maxwell's equation in Riemannian space. J. Sci. Hirosima Univ. Ser. A. **11**, 293–296 (1942).

Maxwell's equation in Riemannian space is obtained by generalizing Rumer's equation [Z. Physik **65**, 244–252 (1930)]. It is pointed out that this equation, recast in tensor form, is similar to the one obtained by K. Morinaga [J. Sci. Hirosima Univ. Ser. A. **5**, 151–188 (1935)].
                                    C. Kikuchi (East Lansing, Mich.).

Cap, F. Zum zweidimensionalen Feldproblem zweier leitenden Ebenen in beliebiger Lage. Österreich. Ing.-Arch. **2**, 201–211 (1948).

The problem considered is the determination of the electric field between two arbitrarily oriented infinitely long conducting plates of finite width. A solution is first obtained for the case in which the plates are parallel nonoverlapping strips with a common axis by means of the Schwarz-Christoffel transformation. If the plates are of equal width the resulting integral can be evaluated in closed form; otherwise series expansions must be used. The general solution for arbitrary orientations is obtained by rotating one strip through a suitably chosen angle.   *M. C. Gray*.

Landsberg, Max. Ein Minimalproblem als Grundlage für die Berechnung von Kabelkapazitäten. Z. Angew. Math. Mech. **28**, 143–152 (1948).

The operating capacity of more than two cables is calculated with the aid of minimum principles. With certain modifications of these principles, the centers of electricity can also be computed.   *A. E. Heins* (Pittsburgh, Pa.).

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Articles of mathematical interest contained in these volumes are as follows: I. M. Vinogradov, Additive problems of the theory of prime numbers; N. G. Čebotarev, The problem of resolvents; M. A. Lavrent'ev, The theory of quasiconformal mapping; S. N. Bernšteĭn, On the role of inequalities and extremal problems in mathematical analysis; P. S. Aleksandrov, Duality theorems in combinatorial topology; L. A. Lyusternik, The theorem about three geodesics; V. I. Smirnov, The works of V. A. Steklov on expansions in orthogonal functions; I. G. Petrovskiĭ, On the theory of partial differential equations; N. M. Krylov,·On certain directions in the domain of approximate solution of the problems of mathematical physics; A. N. Kolmogorov, The statistical theory of oscillations with a continuous spectrum; N. D. Papaleksi, Nonlinear oscillations; I. A. Kibel', The prediction of the weather as a problem in dynamical meteorology; V. V. Golubev, N. E. Žukovskiĭ and contemporary technical aeromechanics; A. I. Nekrasov, The works of S. A. Čaplygin on aerodynamics; Yu. A. Šimanskiĭ, The works of A. N. Krylov in the domain of the oscillation of a ship in the sea; V. V. Sokolovskiĭ and G. S. Šapiro, The methods of B. G. Galerkin in the theory of elasticity; I. I. Artobolevskiĭ, A survey of Soviet science in the development of the theory of the structure of mechanisms; M. V. Kurpičev, Thermal modelling; L. S. Leĭbenson, Underground hydrogasodynamics.

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Volume 1, number 1, appeared in 1948. The journal is published in Osaka.

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Volume 1, number 1, is dated March, 1948. The journal is published by the Oxford University Press.